Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing **Solutions KW38** **HS16**

Solution 1: Dependency

(a) Since X_1 and X_2 are independent, we have

$$\begin{aligned} \Pr[X_1 + X_2 = 8] &= \sum_{x_1=1}^{6} \Pr[X_1 = x_1 \wedge X_2 = 8 - x_1] \\ &= \sum_{x_1=1}^{6} \Pr[X_1 = x_1] \cdot \Pr[X_2 = 8 - x_1] \\ &= 5 \cdot \frac{1}{36}, \end{aligned}$$

where we have used that $\Pr[X_2 = 7] = 0$, while all other probabilities are $1/6$.

(b) By the definition of conditional probability, we have

$$\Pr[X_1 + X_2 \geq 6 \mid X_1 \leq 2] = \frac{\Pr[X_1 + X_2 \geq 6 \wedge X_1 \leq 2]}{\Pr[X_1 \leq 2]} =$$

$$= \frac{\Pr[X_1 = 1] \cdot \sum_{i=5}^{6} \Pr[X_2 = i] + \Pr[X_1 = 2] \cdot \sum_{i=4}^{6} \Pr[X_2 = i]}{\Pr[X_1 = 1] + \Pr[X_1 = 2]} = \frac{\frac{1}{6} \cdot \frac{2}{6} + \frac{1}{6}\frac{3}{6}}{\frac{1}{6} + \frac{1}{6}} = \frac{5}{12}.$$

(c) In each of the following cases, we calculate (by manually counting how many of the 36 possible outcomes are contained in the events) the values of $p_1 = \Pr[\mathcal{E}_1]$, $p_2 = \Pr[\mathcal{E}_2]$, and $p_{12} = \Pr[\mathcal{E}_1 \cap \mathcal{E}_2]$, respectively. The events are then independent iff $p_1 p_2 = p_{12}$.

(i)
- $p_1 = 1/2$.
- $p_2 = 1/2$.
- $p_{12} = 1/4$.
- independent

(ii)
- $p_1 = 1/2$.
- $p_2 = 5/12$.
- $p_{12} = 3/12$.
- dependent

(iii)
- $p_1 = 1/6$.
- $p_2 = 1/6$.
- $p_{12} = 1/18$.
- dependent

(iv)
- $p_1 = 7/12$.
- $p_2 = 1/12$.
- $p_{12} = 1/18$.
- dependent

Solution 2: Geometric Distributions

(a) Let X be the random variable that describes the number of runs until we encounter the first success. For example, abbreviating 'failure' by F and 'success' by S, if we encounter the sequence FFS then X would assume the value 3. The distribution of X is given by

$$\Pr[X = k] = (1-p)^{k-1}p \quad (k = 1, 2, \dots).$$

As we remember from the course *Probability and statistics*,

$$\mathbf{E}[X] = \frac{1}{p}.$$

If we don't remember then we can also compute it like this:

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{k=1}^{\infty} \Pr[X \geq k] \\
&= \sum_{k=1}^{\infty} (1-p)^{k-1} \\
&= \frac{1}{1-(1-p)} \quad \text{(geometric series!)} \\
&= \frac{1}{p}.
\end{aligned}$$

(b) We sum over all even values for X and obtain (using the standard formula for geometric series)

$$\Pr[X \text{ even}] = \sum_{j=1}^{\infty} \Pr[X = 2j] = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{2j} = \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{1}{4}\right)^{j} = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{3}.$$

(c) According to the same calculation,

$$\Pr[X \text{ even}] = \sum_{j=1}^{\infty} \Pr[X = 2j] = \sum_{j=1}^{\infty} (1-p)^{2j-1} p$$
$$= p(1-p) \sum_{j=0}^{\infty} \left((1-p)^2\right)^j = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

Solution 3: Expected running time

(a) Applying the definition of expected value,

$$\mathbf{E}[X] = \sum_{x=1}^{3} \Pr[X = x] \cdot x = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 = \frac{7}{4}.$$

Likewise,

$$\mathbf{E}\left[X^2\right] = \sum_{x=1}^{3} \Pr[X = x] \cdot x^2 = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 9 = \frac{15}{4}.$$

Thus we are reminded that the numbers $\mathbf{E}\left[X^2\right]$ and $\mathbf{E}[X]^2$ are, in general, not equal. Indeed our example has $\mathbf{E}\left[X^2\right] = \frac{15}{4}$, but $\mathbf{E}[X]^2 = \frac{49}{16}$.

(b) The random variables X_1, X_2 are independent and have the same distribution as X. (Note: This is only true because of the specific way the question is phrased. In general we have to be careful whether our random variables are really independent.)

For (i), by applying linearity of expectation, we get

$$\mathbf{E}[X_1 + X_2] = \mathbf{E}[X_1] + \mathbf{E}[X_2] = 2\,\mathbf{E}[X] = \frac{7}{2}.$$

Note that this would hold true even if X_1 and X_2 were dependent. For (ii), on the other hand, we use that X_1,X_2 are independent, and then

$$\mathbf{E}[X_1 \cdot X_2] = \mathbf{E}[X_1] \cdot \mathbf{E}[X_2] = \mathbf{E}[X]^2 = \frac{49}{16}.$$

For (iii), to get a sum of $X_1 + X_2 \leq 4$, there are only the following possibilities:

- $X_1 = 1$,
- $X_1 = 2$ and $X_2 \in \{1, 2\}$, or
- $X_1 = 3$ and $X_2 = 1$.

Since these three events are disjoint, we find

$$\Pr[X_1 + X_2 \le 4] = \Pr[X_1 = 1] + \Pr[X_1 = 2 \text{ and } X_2 \in \{1,2\}] + \Pr[X_1 = 3 \text{ and } X_2 = 1].$$

Since X_1 and X_2 are independent, we obtain

$$\begin{aligned}\Pr[X_1 + X_2 \le 4] &= \Pr[X_1 = 1] + \Pr[X_1 = 2] \cdot \Pr[X_2 \in \{1,2\}] + \Pr[X_1 = 3] \cdot \Pr[X_2 = 1] \\ &= \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{13}{16}.\end{aligned}$$

(c) It might seem as if the running time is described by the random variable $X \cdot N$, which would lead to the result $\mathbf{E}[X] \cdot \mathbf{E}[N]$ as long as X and N are independent.

However, it is important to note that $X \cdot N$ does *not* describe our situation. It would only be correct to use $X \cdot N$ if, for some reason, every subroutine call had the exact same running time. (Why?)

The correct way to express the overall running time is to use a sequence of random variables $X_1, \ldots, X_N$, where X_i describes the running time of the ith subroutine call. In order to be able to compute with the strange formula $X_1 + \cdots + X_N$, we actually use an infinite sequence of variables $X_1, X_2, \ldots$, where the variable X_i is defined to assume the value 0 whenever $i > N$.

We then have, for all $i \ge 1$:

$$\begin{aligned}\mathbf{E}[X_i \mid i \le N] &= \mathbf{E}[X], \\ \mathbf{E}[X_i \mid i > N] &= 0.\end{aligned}$$

Now we can calculate

$$\begin{aligned}\mathbf{E}[X_1 + \ldots X_N] &= \mathbf{E}\left[\sum_{i=1}^{\infty} X_i\right] \\ &= \sum_{i=1}^{\infty} \mathbf{E}[X_i] \quad \text{(by montone convergence)} \\ &= \sum_{i=1}^{\infty} \Big(\underbrace{\mathbf{E}[X_i \mid i \le N]}_{\mathbf{E}[X]} \cdot \Pr[i \le N] + \underbrace{\mathbf{E}[X_i \mid i > N]}_{0} \cdot \Pr[i > N]\Big) \\ &= \mathbf{E}[X] \cdot \sum_{i=1}^{\infty} \Pr[i \le N] \\ &= \mathbf{E}[X] \cdot \mathbf{E}[N].\end{aligned}$$

Solution 4: Random Walks

(a) For any $v \in \{A, B, C, D, E\}$, let us write e_v to denote the expected number of days needed to reach vertex A given that the worm starts from vertex v. The value we are looking for in this task is e_C.

When starting from vertex C, the worm has a probability of 1/2 to go to B in the first step, and a probability of 1/2 to go to D. If it reaches B, it needs another e_B number of days on average to reach A. If it reaches D, it needs e_D expected number of days. Therefore

$$e_C = \frac{1}{2}e_B + \frac{1}{2}e_D + 1.$$

We can write analogous relations for the other quantities:

$$\begin{aligned} e_A &= 0, \\ e_B &= \frac{1}{2}e_A + \frac{1}{2}e_C + 1, \\ e_D &= \frac{1}{2}e_C + \frac{1}{2}e_E + 1, \\ e_E &= \frac{1}{2}e_D + \frac{1}{2}e_A + 1. \end{aligned}$$

This way, we have a linear system of five equations and five unknowns that we can solve. The result is that $e_C = 6$.

(b) According to Markov's inequality, we have

$$\Pr[T \geq 100] = \Pr\left[T \geq \frac{100}{6}e_c\right] \leq \frac{6}{100}.$$

The probability for the worm to take at least 100 days until dinner is at most 6%.

Solution 5: Independence of Three Events

Recall that the events A, B, C are called *pairwise independent* if they satisfy

$$\begin{aligned} \Pr[A \cap B] &= \Pr[A] \cdot \Pr[B], \\ \Pr[A \cap C] &= \Pr[A] \cdot \Pr[C], \\ \Pr[B \cap C] &= \Pr[B] \cdot \Pr[C]; \end{aligned}$$

and they are called *mutually independent* if **in addition** they satisfy

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C].$$

A typical example would be tossing two fair coins. Let A be the event that the first coin lands head. Let B be the event that the second coin lands head. And let C be the event that the two coin tosses land the same. Each event has a probability of $1/2$. By calculating

$$\Pr[A \cap B] = \frac{1}{4},$$
$$\Pr[A \cap C] = \frac{1}{4},$$
$$\Pr[B \cap C] = \frac{1}{4},$$

we see that A, B, C are pairwise independent. But $\Pr[A \cap B \cap C] = 1/4$, not $1/8$ as we would expect for jointly independent events.

Solution 6: Conditional Probability

(a) Intuition: the event that the egg is spoiled is completely independent of the event that the milk is spoiled. Therefore the probability that the egg is spoiled is not influenced by the information that the milk is spoiled.

Formally: Let E be the event that the egg is spoiled and M the event that the milk is spoiled. We are interested in the probability $\Pr[E|M]$. We have

$$\Pr[E|M] = \frac{\Pr[E \cap M]}{\Pr[M]} = \frac{1/4}{1/2} = \frac{1}{2}.$$

(b) Intuition: having exactly one boy and exactly one girl is more likely ($1/2$) than having two boys ($1/4$). Thus if we know that there is at least one boy, it is more likely for the other child to be a girl than that both are boys. Note the important difference to the situation in (a). There, the information we were conditioning on concerned exactly one of the two experiments ("the milk").

In this case, the information concerns both experiments jointly ("one of the two is"). If the information given were that the older child is a boy, then the probability to get another boy would not be influenced by it (given our simplifying assumption on independence).

Formally: Let B be the event that at least one child is a boy and C the event that both children are boys. We are interested in $\Pr[C|B]$.

$$\Pr[C|B] = \frac{\Pr[C \cap B]}{\Pr[B]} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Solution 7: Paradoxes

(a) It is advisable to switch to the other door. Although surprising at first, this is formally verified as follows: Let X be the random variable denoting behind which door the car is. We have $X \in_{\text{u.a.r.}} \{1,2,3\}$. Furthermore, let Y denote the door the showmaster opens. We have

$$Y = \begin{cases} \text{u.a.r. in } \{2,3\} & \text{, if } X = 1 \\ 3 & \text{, if } X = 2 \\ 2 & \text{, if } X = 3 \end{cases}.$$

The information we get is that $Y = 2$. What we want to know is $\Pr[X = 1 \mid Y = 2]$ and $\Pr[X = 3 \mid Y = 2]$. We have

$$\Pr[X = 1 \land Y = 2] = \Pr[X = 1]\Pr[Y = 2 \mid X = 1] = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

$$\Pr[X = 3 \land Y = 2] = \Pr[X = 3]\Pr[Y = 2 \mid X = 3] = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

Furthermore $\Pr[Y = 2] = \frac{1}{2}$, as is easily checked.

Therefore $\Pr[X = 1 \mid Y = 2] = \frac{\Pr[X=1 \land Y=2]}{\Pr[Y=2]} = \frac{1/6}{1/2} = \frac{1}{3}$ and $\Pr[X = 3 \mid Y = 2] = \frac{\Pr[X=3 \land Y=2]}{\Pr[Y=2]} = \frac{1/3}{1/2} = \frac{2}{3}$. Hence in this case we should switch the door, as it doubles the probability that we get the car.

(b) In this case we have

$$\Pr[X = 1 \land Y = 2] = \Pr[X = 1]\Pr[Y = 2 \mid X = 1] = \frac{1}{3} \cdot p = \frac{p}{3},$$

$$\Pr[X = 3 \land Y = 2] = \Pr[X = 3]\Pr[Y = 2 \mid X = 3] = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

We calculate by the law of total probability that

$$\Pr[Y = 2]$$

$$= \Pr[Y = 2 \mid X = 1]\Pr[X = 1] + \Pr[Y = 2 \mid X = 2]\Pr[X = 2] + \Pr[Y = 2 \mid X = 3]\Pr[X = 3]$$

$$= p \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1+p}{3}.$$

Therefore $\Pr[X = 1 \mid Y = 2] = \frac{\Pr[X=1 \land Y=2]}{\Pr[Y=2]} = \frac{p/3}{(1+p)/3} = \frac{p}{1+p}$ and $\Pr[X = 3 \mid Y = 2] = \frac{\Pr[X=3 \land Y=2]}{\Pr[Y=2]} = \frac{1/3}{(1+p)/3} = \frac{1}{1+p}$.

As always $\frac{p}{1+p} \le \frac{1}{1+p}$, we should switch the door also in this setting. Only if $p = 1$, it acutually does not matter what we do.

The case when the show master opens door 3 is completely symmetric; we would need to replace p by $(1 - p)$ in the calculations.

Therefore we have shown that in both cases switching is an optimal strategy. What is now the probability to get the car when always switching? Since the probability that the car is behind door 1 is $\frac{1}{3}$, and we get the car exactly if it is *not* behind door 1, this probability is $\frac{2}{3}$, as in (a).

(c) There is no reason to switch, it does not matter. After Hermione has died, Harry can exclude only one of the three possible locations of the powerful potion. The other two locations are still equally likely, so he has a probability of 1/2 to fetch the right one.

What is the difference to (a)? Here we have exactly the alternative situation we described before. Since Hermione selects and inspects one of the two other potions *at random*, the information collected is simply "the good potion was *not there*". Hermione cannot give away any information beyond what she collects by her random experiment because she is not in possession of any such information.