Solution 1: Finding a Separating Line

Our linear program has variables $a$, $b$, $c$, $\varepsilon$ and looks as follows (the number of constraints is $|R| + |B| + 1$):

$$\begin{align*}
\text{maximize } & \varepsilon \\
\text{subject to } & ax + by \geq c + \varepsilon \quad (\text{for each } (x, y) \in R), \\
& ax + by \leq c - \varepsilon \quad (\text{for each } (x, y) \in B), \\
& \varepsilon \leq 1.
\end{align*}$$

We will show the following (assuming $R, B \neq \emptyset$):

1. Our LP has an optimal solution.
2. If some optimal solution $(a^*, b^*, c^*, \varepsilon^*)$ satisfies $\varepsilon^* > 0$ and $(a^*, b^*) \neq (0, 0)$, then there is a separating line, namely the line $a^*x + b^*y = c^*$.
3. Conversely, if there is a separating line, then every optimal solution $(a^*, b^*, c^*, \varepsilon^*)$ satisfies $\varepsilon^* > 0$ and $(a^*, b^*) \neq (0, 0)$.

From this it will be clear how to decide, given an optimal solution, if a separating line exists (namely, check if $\varepsilon^* > 0$ and $(a^*, b^*) \neq (0, 0)$) and, if so, how to compute one (take the line $a^*x + b^*y = c^*$).

Ad 1: Our LP is feasible, because the origin is a feasible point. Furthermore the objective function is bounded, because we artificially bounded it with the constraint $\varepsilon \leq 1$. Hence our LP has an optimal solution.

Ad 2: Clear, because by construction any feasible point $(a, b, c, \varepsilon)$ with $\varepsilon > 0$ and $(a, b) \neq (0, 0)$ defines a separating line $ax + by = c$.

Ad 3: Assume there exists a separating line $\ell : a_0x + b_0y = c_0$, where we choose the coefficients in a normalized way such that $a_0^2 + b_0^2 = 1$ and such that $a_0x + b_0y > c_0$ for all $(x, y) \in R$. Let $\varepsilon_0 := \min\{1, \text{dist}(\ell, R), \text{dist}(\ell, B)\}$. Then $(a_0, b_0, c_0, \varepsilon_0)$ is feasible with $\varepsilon_0 > 0$. This shows that the optimal value (which we already know to exist) is positive: $\varepsilon^* \geq \varepsilon_0 > 0$. It remains to show $(a^*, b^*) \neq (0, 0)$. Assume otherwise. Then $c^* + \varepsilon^* \leq 0 \leq c^* - \varepsilon^*$ implies $\varepsilon^* \leq 0$, a contradiction to what we proved a moment ago.
Solution 2: Fitting a Ball into a Convex Polytope

We assume that the given intersection of halfspaces is bounded, so that the question is well-defined and fits the title of the exercise. (This was an oversight in phrasing the question.)

We may assume without loss of generality that every constraint $a_i^T x \leq b_i$ (which corresponds to the halfspace $H_i$) is normalized, which simply means that we have $\|a_i\|_2 = 1$. Indeed, this can always be achieved by rescaling all the coefficients in a given constraint.

From linear algebra we recall that $a_i$ is nothing else than the normal vector of the hyperplane $P_i$ defined by the equation $a_i^T x = b_i$ (note that $P_i$ is simply the boundary of $H_i$). Furthermore, the absolute value of $b_i$ is equal the Euclidean distance between $P_i$ and the origin (note that this is only true because we have normalized constraints). The sign of $b_i$ additionally tells us on which side of the coordinate system $P_i$ lies relative to $a_i$. If $b_i = 0$ then $P_i$ goes through the origin. If $b_i > 0$ then $P_i$ has been moved away from the origin in the direction of the normal vector $a_i$. If $b_i < 0$ then $P_i$ has been moved away from the origin against the direction of $a_i$.

Let us now define for every halfspace $H_i$ another halfspace $H'_i := \{x: a_i^T x \leq b_i - r\}$ for a non-negative parameter which we call $r$. We also define the corresponding hyperplanes $P'_i$. Note that for $r > 0$, $H'_i$ is smaller than $H_i$ in the sense that it is a strict subset. More precisely, the boundary $P'_i$ of $H'_i$ has been moved inwards by a distance of $r$ when compared with the boundary $P_i$ of $H_i$.

Now suppose $r \geq 0$ and that $H' := \bigcap_{i=1}^m H'_i$ is non-empty. Fix any point $c$ in $H'$. Clearly, the ball with center point $c$ and radius $r$ must be completely contained in $H := \bigcap_{i=1}^m H_i$ because $c$ has distance at least $r$ from $P_i$ for all indices $i$. Conversely, for any given ball with center point $c$ and radius $r$ that is completely contained in $H$, it must also be the case that $c$ is contained in $H_i$ for otherwise the given ball would properly intersect one of the hyperplanes $P_i$. Therefore, the desired largest radius $r'$ is equal to the largest value of $r \geq 0$ with $H' \neq \emptyset$, and the desired center point $c'$ can be any point in $H''$. All of the above conditions can easily be expressed in the following linear program with real variables $c \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

Maximize $r$

subject to $r \geq 0$

\[ a_1^T c \leq b_1 - r \]
\[ a_2^T c \leq b_2 - r \]
\[ \vdots \]
\[ a_m^T c \leq b_m - r \]
Solution 3: Linear Programs in Equational Form

Given a linear program in standard form\(^1\)

\[
\text{maximize } c^T x \text{ subject to } Ax \leq b, \tag{LP 1}
\]

where \(A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n\) and \(b \in \mathbb{R}^m\), we can convert it into equational form as follows:

First we replace the “\(\leq\)” by a “\(=\)” by the following trick.

\[
\text{maximize } c^T x \text{ subject to } Ax + \varepsilon = b, \varepsilon \geq 0 \tag{LP 2}
\]

where \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)\) is a vector of \(m\) new variables. In a second step we get rid of unconstrained (= possibly negative) variables. To this end we replace each \(x_i\) by \(x'_i - x''_i\), where \(x'_i, x''_i\) are two new nonnegative variables:

\[
\text{maximize } c^T(x' - x'') \text{ subject to } A(x' - x'') + \varepsilon = b, \ x' \geq 0, \ x'' \geq 0, \ \varepsilon \geq 0. \tag{LP 3}
\]

Now we write this LP in such a way that it is undoubtedly in equational form, e.g. like this:

\[
\begin{pmatrix}
\frac{c}{-c} \\
0
\end{pmatrix}^T
\begin{pmatrix}
x' \\
x'' \\
\varepsilon
\end{pmatrix}
\text{ subject to } \begin{pmatrix}
A & -A & I_m
\end{pmatrix}
\begin{pmatrix}
x'
\varepsilon
\end{pmatrix} = b, \begin{pmatrix}
x'
\varepsilon
\end{pmatrix} \geq 0. \tag{LP 4}
\]

In what sense have we “converted” the LP into equational form? We make the following notes.

- If \(x\) is a feasible solution of (LP 1), then a corresponding feasible solution of (LP 4) is given by

\[
\begin{align*}
x' &= \max(x_1, 0), \ldots, \max(x_n, 0), \\
x'' &= -(\min(x_1, 0), \ldots, \min(x_n, 0)), \\
\varepsilon &= b - Ax.
\end{align*}
\]

Furthermore this feasible solution to (LP 4) has the same objective value as \(x\).

- Correspondingly, if \((x', x'', \varepsilon)\) is a feasible solution of (LP 4), then \(x' - x''\) is a feasible solution of (LP 1) with the same objective value.

Complexity: The linear program (LP 4) has \(2n + m\) variables and \(2n + 2m\) constraints, where the original (LP 1) had \(n\) variables and \(m\) constraints.

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\(^1\)You might want to revise at this point how to convert any linear program into standard form (section 6.1 of the lecture notes).
Solution 4: Maximum Number of Vertices of 3-dimensional Convex Polytopes

The vertex-edge graph of $P$ is a planar graph. (Why? Think of cutting a small hole into some face of $P$ and then stretching the whole thing flat.) For the number of faces, $f$, we have $f \leq n$ because each of our halfspaces defines at most one (unique) face of the polytope. In order to bound the number of edges, $e$, we count the vertex-edge incidences in two ways:

$$3v \leq \# \{(\xi, \eta) : \xi \text{ is a vertex, } \eta \text{ is an edge that is incident to } \xi \} = 2e.$$ 

Thus $e \geq \frac{3}{2}v$. Plugging this into Euler’s formula we find $2 = v - e + f \leq v - \frac{3}{2}v + f = f - \frac{v}{2}$, which gives $v \leq 2f - 4 \leq 2n - 4$.

Solution 5: Certificates for Infeasibility of Systems of Linear Equations

Let us first do the easy direction. Suppose there is $y$ with $A^Ty = 0$ and $b^Ty = 1$. Furthermore, towards a contradiction, suppose there is an $x$ with $Ax = b$ (or, equivalently, $x^TA^T = b^T$). We arrive at a contradiction (and hence conclude that $Ax = b$ is unsolvable) by observing that

$$0 = x^T0 = x^TA^Ty = b^Ty = 1.$$ 

For the other direction we recall some notation from linear algebra (we assume throughout that the matrix $A$ has $m$ rows and $n$ columns). The image of $A$ is the set $\text{img}(A) := \{Ax | x \in \mathbb{R}^n\}$. The left nullspace (or cokernel) of $A$ is the set $\ker(A^T) := \{y \in \mathbb{R}^m | A^Ty = 0\}$. We also recall that these two sets are vector spaces and that they are orthogonal complements of each other. In particular, if $i_1, \ldots, i_r$ is an orthonormal basis of $\text{img}(A)$ and $k_1, \ldots, k_s$ is an orthonormal basis of $\ker(A^T)$, then $i_1, \ldots, i_r, k_1, \ldots, k_s$ is an orthonormal basis of $\mathbb{R}^m$.

Now suppose that the system $Ax = b$ is unsolvable. We show how to construct $y$ with $A^Ty = 0$ and $b^Ty = 1$. First we write $b$ as a linear combination $b = \alpha_1 i_1 + \cdots + \alpha_r i_r + \beta_1 k_1 + \cdots + \beta_s k_s$. We observe that $s \geq 1$ and that for some index $i$ we must have $\beta_i \neq 0$ (for otherwise $b \in \text{img}(A)$, which cannot be if $Ax = b$ is unsolvable). W.l.o.g. we assume that $\beta_1 \neq 0$ and we define $y := \frac{1}{\beta_1}k_1$. We now see that $A^Ty = 0$ because $y \in \ker(A^T)$. Moreover,

$$b^Ty = \frac{\alpha_1}{\beta_1} i_1^T k_1 + \cdots + \frac{\alpha_r}{\beta_1} i_r^T k_1 + \frac{\beta_1}{\beta_1} k_1^T k_1 + \frac{\beta_2}{\beta_1} k_1^T k_1 + \cdots + \frac{\beta_s}{\beta_1} k_1^T k_1 = \frac{\beta_1}{\beta_1} = 1.$$ 

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