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Solution 1: Strong Duality

1. Neither (P) nor (D) has a feasible solution: An example for this first case can be seen directly below. We observe that neither linear program is feasible because in (P) the constraint $x_1 \leq -1$ contradicts non-negativity of x_1, and in (D) the constraint $-y_2 \geq 1$ contradicts non-negativity of y_2.

$$\begin{aligned} (P):\quad & \text{Maximize } x_1 + x_2 \\ & \text{subject to } x_1, x_2 \geq 0 \\ & x_1 \leq -1 \\ & -x_2 \leq -1 \end{aligned} \qquad \begin{aligned} (D):\quad & \text{Minimize } -y_1 - y_2 \\ & \text{subject to } y_1, y_2 \geq 0 \\ & y_1 \geq 1 \\ & -y_2 \geq 1 \end{aligned}$$

2. (P) is unbounded and (D) has no feasible solution: An example for this second case can again be seen below. We observe that (P) is indeed unbounded because we can put $x_1 := 1$ and, at the same time, make x_2 arbitrarily large, which makes also the objective function arbitrarily large. On the other hand, (D) is infeasible because the constraint $-y_2 \geq 1$ directly contradicts non-negativity of y_2.

$$\begin{aligned} (P):\quad & \text{Maximize } x_1 + x_2 \\ & \text{subject to } x_1, x_2 \geq 0 \\ & x_1 \leq 1 \\ & x_1 - x_2 \leq -1 \end{aligned} \qquad \begin{aligned} (D):\quad & \text{Minimize } y_1 - y_2 \\ & \text{subject to } y_1, y_2 \geq 0 \\ & y_1 + y_2 \geq 1 \\ & -y_2 \geq 1 \end{aligned}$$

3. (P) has no feasible solution and (D) is unbounded: For this case we simply reverse the roles of the two linear programs from the previous case. Note that in the process we change the names of the variables and we multiply the constraints and objective functions by -1 in order to stay true to the schema from the lecture notes.

$$\begin{aligned} (P):\quad & \text{Maximize } -x_1 + x_2 \\ & \text{subject to } x_1, x_2 \geq 0 \\ & -x_1 - x_2 \leq -1 \\ & x_2 \leq -1 \end{aligned} \qquad \begin{aligned} (D):\quad & \text{Minimize } -y_1 - y_2 \\ & \text{subject to } y_1, y_2 \geq 0 \\ & -y_1 \geq -1 \\ & -y_1 + y_2 \geq 1 \end{aligned}$$

4. Both (P) and (D) have a feasible solution: Depicted below is a linear program (P) and its dual (D). On one hand, $\mathbf{x}^* = (1, 2)$ is a feasible solution of (P) with objective value $1 + 2 = 3$. On the other hand, $\mathbf{y}^* = (0.5, 0.5)$ is a feasible solution of (D) with objective value $4 \cdot 0.5 + 2 \cdot 0.5 = 3$. So, clearly, both (P) and (D) are feasible. Additionally, since

the objective values of x^* and y^* are identical, weak duality tells us that x^* and y^* must in fact be optimal solutions of the respective linear programs.

$$\begin{aligned} \text{(P):} \quad & \text{Maximize } x_1 + x_2 \\ \text{subject to } & x_1, x_2 \geq 0 \\ & 2x_1 + x_2 \leq 4 \\ & x_2 \leq 2 \end{aligned} \qquad \begin{aligned} \text{(D):} \quad & \text{Minimize } 4y_1 + 2y_2 \\ \text{subject to } & y_1, y_2 \geq 0 \\ & 2y_1 \geq 1 \\ & y_1 + y_2 \geq 1 \end{aligned}$$

Solution 2: The Subtour LP

(i) "$\Rightarrow$": Assume that G is connected, and let $S \subseteq V$, $\emptyset \neq S \neq V$. Then $(S, V \setminus S)$ is a partition of V into two nonempty subsets. The set $\delta(S)$ consists exactly of those edges that have one endpoint in S and the other one in $V \setminus S$. If this set were empty then G would be disconnected (because there could be no path from any vertex in S to any vertex in $V \setminus S$).

"$\Leftarrow$": Let us write "condition $(*)$" for the condition $\delta(S) \neq \emptyset$ for all $S \subseteq V$, $\emptyset \neq S \neq V$. Assume that G satisfies condition $(*)$. Let $v \in V$. We want to show that there is a path from v to every other vertex. To this end let C_v denote the connected component of v in G. Applying condition $(*)$ to C_v, there are only three possibilities: (1) There is an edge from C_v to $V \setminus C_v$; but this contradicts the definition of a connected component. (2) $C_v = \emptyset$; but this contradicts $v \in C_v$. (3) $C_v = V$, q.e.d.

(ii) Let G be any graph that has an isolated vertex. Then the subtour LP includes a constraint of the form $0 = 2$, and is clearly infeasible.

(iii) Let G be the Petersen graph. It is known that G is non-Hamiltonian, so that the Subtour LP cannot have an integer solution. It is also known (or obvious) that G is 3-connected as well as 3-regular. From this it follows that setting $c_e = 2/3$ for all edges e gives a feasible solution to the Subtour LP.

(If you have not seen the Petersen graph before, a websearch will give you many pictures of it. It happens to be one of the smallest non-Hamiltonian graphs out there, which should explain why we use it for this question.)

(iv) Let x be a feasible point and assume that there is $\eta \in E$ with $x_\eta > 1$. Write $\eta = \{u, v\}$ (the two endpoints of the bad edge). The first contraint of our LP reads $\sum_{e \in \delta(v)} x_e = \sum_{e \in \delta(u)} x_e = 2$. In other words, $\sum_{e \in \delta(\{u,v\})} x_e = 4 - 2x_\eta < 2$, a contradiction to the second constraint applied to $S := \{u, v\}$. (In the last step we have used the assumption $|V| > 3$, which guarantees $S \neq V$.)

Solution 3: The Loose Spanning Tree LP

We have seen in lemma 6.10 that the optimal value is at most $\frac{\ell\gamma}{2}$, and we want to show that it cannot be smaller than that. Let x be a feasible solution. Its value $c^T x$ depends only on the values x_e along the path (because the edges of the clique all have weight 0). If those values are all $x_e = \frac{1}{2}$, then we get said value $\frac{\ell\gamma}{2}$. Assume that there is some edge η along the path with

$x_\eta = \frac{1}{2} - \varepsilon$, $\varepsilon > 0$. Then the second constraint of the LP implies that *all* other edges e along the path have $x_e \geq \frac{1}{2} + \varepsilon$. The resulting sum is $c^T x \geq \gamma(\frac{1}{2} - \varepsilon) + (\ell - 1)\gamma(\frac{1}{2} + \varepsilon) > \frac{\ell\gamma}{2}$.

Solution 4: Loose and Tight Spanning Tree LP

Let x be a feasible point of the Tight Spanning Tree LP. Let $S \subseteq V$, $\emptyset \neq S \neq V$. We want to show that x also satisfies the constraint

$$\sum_{e \in \delta(S)} x_e \geq 1. \tag{$*$}$$

We know

$$\sum_{e \in E \cap \binom{S}{2}} x_e \leq |S| - 1,$$
$$\sum_{e \in E \cap \binom{V \setminus S}{2}} x_e \leq |V \setminus S| - 1,$$

which together is at most $|V| - 2 = n - 2$. Due to the first constraint, $\sum_{e \in E} x_e = n - 1$, the values x_e of the remaining edges (those not in $\binom{S}{2}$ or $\binom{V \setminus S}{2}$) must sum up to at least 1. This is exactly the statement $(*)$ that we wanted to show.

Solution 5: Let's Relax

Let $S = \{s_1, \ldots, s_n\}$, and let the numbering be chosen in such a way that $\min_{x \in S} c^T x = c^T s_1$. Let $y \in \text{conv}(S)$. By definition of the convex hull there is $\lambda \in \mathbf{R}^n$ such that $\lambda \geq 0$, $\mathbf{1}_n^T \lambda = 1$ and $y = \sum_{i=1}^n \lambda_i s_i$. We find

$$c^T y = \sum_{i=1}^n \lambda_i c^T s_i \geq \sum_{i=1}^n \lambda_i c^T s_1 = \left(\min_{x \in S} c^T x\right) \sum_{i=1}^n \lambda_i = \min_{x \in S} c^T x.$$

Since $y \in \text{conv}(S)$ was arbitrary, we obtain

$$\min_{y \in \text{conv}(S)} c^T y \geq \min_{x \in S} c^T x$$

and "$\leq$" clearly holds anyways.