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(a) Let M be any Olympic matching, and let T be any Olympic transversal. Since T contains by definition an element (colour) of every flag, it must contain an element $c^{(f)} \in f$ for each $f \in M$. Since the flags $f \in M$ are pairwise disjoint, the colours $c^{(f)}$ must be pairwise distinct. Thus we have shown that T must contain at least as many colours as there are flags in M, that is, $|T| \geq |M|$. Since this reasoning was valid for arbitary M and T, it implies $\tau(F) \geq \mu(F)$.

(b) The linear program (LP-T) is feasible, because $\mathbf{1}_n$ is a feasible point. Furthermore its objective function is bounded from below, because for all feasible x we have $\mathbf{1}_n^T x = \sum_{i=1}^n x_i \geq 0$. From these two facts it follows that there is a (finite) optimum.

Now let T be an Olympic transversal of minimum size, $|T| = \tau(F)$. Then the (characteristic) vector $x \in \{0,1\}^n$ with

$$x_i = \begin{cases} 0 & \text{if } c_i \notin T \\ 1 & \text{if } c_i \in T \end{cases}$$

is feasible with objective value $\mathbf{1}_n^T x = |T|$. The optimum of the linear program can only be smaller; hence $\tau^*(F) \leq \tau(F)$.

(c) Let $n = 3$ and $F = \big\{\{c_1, c_2\}, \{c_2, c_3\}, \{c_1, c_3\}\big\}$ (a triangle graph). Here every Olympic transversal must contain at least two colours, and $T = \{c_1, c_2\}$ is an Olympic transversal; so we have $\tau(F) = 2$. On the other hand, $x = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is feasible, so we have $\tau^*(F) \leq \frac{3}{2}$.

(d) Consider the dual linear program,

$$\text{maximize } \mathbf{1}_m^T y \text{ subject to } y \geq 0,\ A^T y \leq \mathbf{1}_n. \qquad \text{(LP-M)}$$

Since (LP-T) has an optimal solution, so also (LP-M) has an optimum $\mu^*(F)$.

Let M be an Olympic matching of maximum size, $|M| = \mu(F)$, and let $y \in \{0,1\}^m$ denote its characteristic vector, that is,

$$y_i = \begin{cases} 0 & \text{if } f_i \notin M \\ 1 & \text{if } f_i \in M. \end{cases}$$

Then y is a feasible solution for (LP-M) with objective value $|M|$, thus $|M| \leq \mu^*(F)$. Since M was an arbitrary Olympic matching, this proves $\mu(F) \leq \mu^*(F)$. By weak duality we have $\mu^*(F) \leq \tau^*(F)$, and the statement follows.

(e) Let $T = \{c_1, \dots, c_s\}$, and let N denote the number of flags not intersected by T. We have

$$\mathbf{E}[N] = \mathbf{E}\left[\sum_{i=1}^{m} [\, f_i \cap T = \emptyset \,]\right] = \sum_{i=1}^{m} \Pr[\, f_i \cap T = \emptyset \,]$$

and for all i we have

$$\begin{aligned}
\Pr[\, f_i \cap T = \emptyset \,] &= \Pr[\, c_1 \notin f_i \text{ and } \dots \text{and } c_s \notin f_i] \\
&= \prod_{k=1}^{s} \Pr[\, c_k \notin f_i \,] \qquad \text{(the draws are independent)} \\
&= \prod_{k=1}^{s} (1 - \Pr[\, c_k \in f_i \,]) \\
&= \prod_{k=1}^{s} \left(1 - \sum_{c \in f_i} p(c)\right) \\
&= \prod_{k=1}^{s} \left(1 - \frac{\sum_{c_j \in f_i} x_j^*}{\sum_{j=1}^{n} x_j^*}\right) \\
&\le \prod_{k=1}^{s} \left(1 - \frac{1}{\sum_{j=1}^{n} x_j^*}\right) \qquad \text{(because } x^* \text{ is feasible for (LP-T))} \\
&= \prod_{k=1}^{s} \left(1 - \frac{1}{\tau^*}\right) = \left(1 - \frac{1}{\tau^*}\right)^s \le e^{-s/\tau^*}.
\end{aligned}$$

It follows that $\mathbf{E}[N] \le m e^{-s/\tau^*}$. If we choose $s := \lfloor \tau^* \ln m \rfloor + 1$ (note that we must choose an integer number here) then we have $s > \tau^* \ln m$ and we obtain

$$\mathbf{E}[N] < m e^{-(\tau^* \ln m)/\tau^*} = \frac{m}{m} = 1.$$

It follows that the random set T is an Olympic transversal with positive probability (because otherwise we would have $\Pr[N \ge 1] = 1$ and hence $\mathbf{E}[N] = \sum_{k \ge 1} \Pr[N \ge k] \ge 1$). This means, concretely, that there is a fixed choice of colours $c_1, \dots, c_s$ such that T is an Olympic transversal. Since $|T| \le s$ this proves $\tau(F) \le s \le \tau^* \ln m + 1$.

(f) (i) Assume w.l.o.g that the colours are numbered in such a way that the first, say, d colours are dark and the remaining colours are light. The assumption in the question means, in terms of the incidence matrix A, that every row has exactly two entries equal to 1, namely one entry within the first d columns and another entry within the remaining $n - d$ columns. All other entries are zero. It follows that the sum of the first d columns of A equals $\mathbf{1}_m$, and the sum of the remaining columns is also $\mathbf{1}_m$. This is a linear dependence among the columns of A, which for clarity we can also write as

$$A \begin{pmatrix} \mathbf{1}_d \\ -\mathbf{1}_{n-d} \end{pmatrix} = 0.$$

(ii) We prove by induction on $k \geq 1$ that for of every $k \times k$-submatrix B of A we have $\det(B) \in \{-1, 0, 1\}$.

Induction base case, $k = 1$*:* Clear because the entries of A are all $\in \{0, 1\}$.

Induction step: Let $k \geq 2$ and assume that the statement holds for all smaller square submatrices. Let B be a $k \times k$-submatrix of A. If every row of B contains two non-zero entries, then we can prove in the same way as in (i) that the columns of B are linearly dependent, from which it follows that $\det(B) = 0$.

Now assume the opposite, that there is a row i of B that has at most one non-zero entry. If all entries in this row are zero, then we again have $\det(B) = 0$. So now assume that row i contains exactly one non-zero entry, say in column j. Then $B_{ij} = 1$. Let C denote the matrix that arises from B by deleting the ith row and the jth column. From the Laplace expansion for the determinant (or directly from the formula for the determinant, if you're patient enough to go through this) it follows that $\det(B) = \pm B_{ij} \det(C) = \pm \det(C)$. Since C is also a submatrix of A, but of dimension smaller than k, we can apply the induction hypothesis and obtain $\det(C) \in \{-1, 0, 1\}$. The statement follows.

(iii) Let x be a basic feasible solution of (LP-T). By definition, x satisfies n linearly independent constraints with equality. If we write the constraints of the linear program in the form

$$Bx \geq b$$

where

$$B := \begin{pmatrix} I_n \\ A \end{pmatrix}, \ b := \begin{pmatrix} \mathbf{0}_n \\ \mathbf{1}_m \end{pmatrix},$$

then the n linearly independent constraints correspond to n linearly independent rows of the matrix B. Let $\tilde{B}$ be the submatrix that consists of these rows, and let $\tilde{b}$ be the vector that contains the corresponding entries of the right-hand side b, so that

$$\tilde{B}x = \tilde{b}.$$

Since the proof in (ii) applies verbatim to the matrix B instead of A, we have $\det(\tilde{B}) \in \{-1, 0, 1\}$. By linear independence, however, we have $\det(\tilde{B}) \neq 0$. By Cramer's rule (cf. the proof of theorem 6.2 where the same trick was used), for all j,

$$x_j = \frac{\det(\tilde{B}_j)}{\det(\tilde{B})} = \pm \det(\tilde{B}_j),$$

where $\tilde{B}_j$ is obtained from $\tilde{B}$ by replacing the jth column with $\tilde{b}$. All entries of $\tilde{B}_j$ are integers, hence $\det(\tilde{B}_j)$ is an integer. Thus we have shown that all entries of x are integers.

(iv) We assume, as we told you by mail you may, too, that (LP-T) has an optimal solution x^* which is at the same time a basic feasible solution. It follows from (iii) that x^* is integral. Furthermore we can observe that every optimal solution satisfies $x^* \leq \mathbf{1}_n$ (if there is some coordinate i with $x_i^* > 1$, then we can replace x_i^* by 1 and the resulting vector is still feasible, but its objective value is strictly smaller, contradicting optimality of x^*). We also have $x^* \geq \mathbf{0}_n$ from the constraints.

Thus x^* must be a 0-1-vector, and the set $T := \{c_j : x_j^* = 1\}$ is an Olympic transversal of minimum size *(by construction of (LP-T); no need to detail this once again for just 2 points)*. It follows that that $\tau^*(F) = |T| = \tau(F)$.

(v) Consider the dual program (LP-M) again. We can write the constraints in the same way as we did for (LP-T) in (iii), i.e.,

$$\begin{pmatrix} -I_m \\ A^T \end{pmatrix} y \leq \begin{pmatrix} \mathbf{0}_m \\ \mathbf{1}_n \end{pmatrix}.$$

Let C be any square submatrix of $\begin{pmatrix} -I_m \\ A^T \end{pmatrix}$. Then, by Laplace expansion, $\det(C) = \pm \det \tilde{C}$ where $\tilde{C}$ is some submatrix of A^T (and also of C). The transposed matrix $\tilde{C}^T$ is a submatrix of A, so we can apply (ii):

$$\det(C) = \pm \det(\tilde{C}) = \pm \det(\tilde{C}^T) \in \{-1, 0, 1\}.$$

This was true for an arbitrary square submatrix C. Hence the reasoning in (iii) can also be applied to the dual program (LP-M), and every basic feasible solution of (LP-M) is integral.

Let y be any feasible solution of (LP-M). We claim that $y \leq \mathbf{1}_m$. Indeed, every column of A^T contains at least one entry (actually, two of them) equal to 1, so that every y_i appears in some constraint of the form $y_{i_1} + \cdots + y_{i_k} \leq 1$. Since we also have $y \geq 0$, this implies $y_i \leq 1$.

We have shown that every basic feasible solution of (LP-M) is a 0-1-vector. Similarly to (iv) it follows that, given an optimal solution y^*, the set $M := \{f_i : y_i^* = 1\}$ is a maximum Olympic matching, and $\mu^*(F) = |M| = \mu(F)$. We conclude with strong duality:

$$\tau(F) = \tau^*(F) = \mu^*(F) = \mu(F).$$