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Algorithms, Probability, and Computing **Special Assignment 2** **HS16**

- The solution is due on **Tuesday December 6** by **4 pm**. Please bring a print-out of your solution with you to the lecture. If you cannot attend (and please only then), you may alternatively send your solution as a PDF, likewise **until 4 pm**, to `mmilatz@inf.ethz.ch`. We will send out a confirmation that we have received your file. Make sure you receive this confirmation within the day of the due date, otherwise complain timely.
- **All further guidelines and requirements on how to solve the exercises are the same as for Special Assignment 1.**

Olympic Colours

$2 + 3 + 2 + 5 + 8 + (2 + 7 + 6 + 2 + 3) = 40$ **points**

The inventor of the Olympic flag, Pierre de Coubertin, wrote in 1931:

> *"The Olympic flag has a white background, with five interlaced rings in the centre: blue, yellow, black, green and red. This design is symbolic; it represents the five continents of the world, united by Olympism, while the six colours are those that appear on all the national flags of the world at the present time."*

This property no longer holds with today's set of national flags: Ireland also uses orange, for instance. But at least it remains true that every country's flag shares some colour with the Olympic flag. We consider the question how many colours would have been necessary to achieve this lesser goal. Naturally, we want to tackle the question in sufficient generality, so as to be prepared for every possible political change. We will also investigate a case where we can find an answer efficiently.[1]

Let $C = \{c_1, \dots, c_n\}$ be a set of colours. Let $F = \{f_1, \dots, f_m\} \subseteq 2^C$ encode all flags, where $f_i \in F$ means that there is some flag that uses exactly the colours given by f_i. We assume that the c_i's and f_i's are numbered in such a way that there are no duplicates (that is, $c_i \neq c_j$ for all $i \neq j$, and similarly $f_i \neq f_j$ for all $i \neq j$). We also assume that every flag uses at least some colour ($f_i \neq \emptyset$ for all i). Finally we also assume $m, n \geq 1$; so, strictly speaking, we are not prepared for every possible political change after all...

The *incidence matrix* of F is the matrix $A \in \{0, 1\}^{m \times n}$ that has $A_{ij} = 1$ if and only if $c_j \in f_i$.

An *Olympic transversal* is a set of colours $T \subseteq C$ that intersects every flag in F. As explained above, we are interested in the minimum size $|T|$ of an Olympic transversal, which we denote by $\tau(F)$.

[1] Speed is important. After all, *"do you not know that the runners in the stadium all run in the race, but only one wins the prize? Run so as to win." (1 Cor 9:24)*

Meanwhile, the choreographer who is in charge of the opening celebrations for the upcoming Olympic games is puzzled over a related problem. He envisages to have exactly one dancer of every colour, and the group of dancers to move around in such a way as to bring out various flags. At every moment each dancer is allowed to participate in at most one flag. Now he would like to know what is the maximum number of flags that can appear on the stage simultaneously. More precisely, if we define an *Olympic matching* to be a subset $M \subseteq F$ of pairwise disjoint flags, then we are interested in the maximum size $|M|$ of an Olympic matching. This maximum size we denote by $\mu(F)$.

(a) Show that $\mu(F) \leq \tau(F)$.
(Please prove this directly, without using later questions and without linear programs.)

(b) We consider the following linear program.

$$\text{minimize } \mathbf{1}_n^T x \text{ subject to } x \geq \mathbf{0}_n,\ Ax \geq \mathbf{1}_m. \qquad \text{(LP-T)}$$

Show that (LP-T) has a finite optimal value $\tau^*(F)$ which satisfies $\tau^*(F) \leq \tau(F)$.

(c) Give an example for a collection of flags F such that $\tau^*(F) < \tau(F)$.

(d) Show that $\tau^*(F) \geq \mu(F)$.

(e) Let x^* be an optimal solution of (LP-T). We define a probability distribution on the set of colours:

$$\begin{array}{rccc} p & : & C & \longrightarrow & [0,1] \\ & & c_i & \longmapsto & \frac{x_i^*}{\sum_{j=1}^n x_j^*}. \end{array}$$

Choose a suitable number s such that the following holds: If we pick colours $c_1, \ldots, c_s$ at random according to p, possibly with repetitions, then the expected number of flags not intersected by $\{c_1, \ldots, c_s\}$ is less than 1. Deduce that $\tau(F) \leq \tau^*(F) \cdot \ln(m) + 1$.

(f) Now we consider a very special competition where the only countries that may participate are those whose flag consists of exactly one dark and exactly one light colour. (Every colour is either dark or light.) Assuming that all $f_i \in F$ satisfy this condition, show the following:

(i) The columns of A are linearly dependent.

(ii) Every square submatrix of A has determinant 0, $+1$, or -1.
Hint: Use induction on the dimension of the submatrix.

(iii) Every basic feasible solution of (LP-T) is integral.

(iv) $\tau^*(F) = \tau(F)$.

(v) $\mu(F) = \tau(F)$.