Chapter 6

Cuboids

We have already seen that we can efficiently find the bounding box $Q(P)$ and an arbitrarily good approximation to the smallest enclosing ball $B(P)$ of a set $P \subseteq \mathbb{R}^d$. Unfortunately, both bounding volumes are bad when the task is to approximate the volume of $\mathrm{conv}(P)$, since the ratios

$$\frac{\mathrm{vol}(Q(P))}{\mathrm{vol}(\mathrm{conv}(P))} \text{ and } \frac{\mathrm{vol}(B(P))}{\mathrm{vol}(\mathrm{conv}(P))}$$

can get arbitrarily large even for $d = 2$, and if $\mathrm{conv}(P)$ has nonzero volume: as the points in P get closer and closer to some fixed—nonvertical and nonhorizontal—line segment, the volume of the convex hull becomes smaller and smaller, while the volumes of $Q(P)$ and $B(P)$ converge to nonzero constants.

In this chapter, we show that boxes of *arbitrary orientations* are better with respect to volume. To distinguish them from the (axis-parallel) boxes, we call them *cuboids*, see Figure 6.1. Formally, a cuboid is any set of the form

$$C = \{Mx \mid x \in Q_d(\underline{b}, \overline{b}), M^{-1} = M^T\}. \tag{6.1}$$

A matrix M with $M^{-1} = M^T$ is called *orthogonal*, and in the orthogonal coordinate system defined by the columns of M, C is an axis-parallel box.

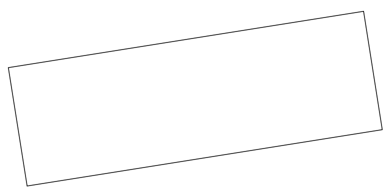

Figure 6.1: A cuboid in $\mathbb{R}^2$

6.1 Approximating the smallest enclosing cuboid

The exact smallest enclosing cuboid $C(P)$ of set $P \subseteq \mathbb{R}^d$ can be computed in time $O(n \log n)$ for $d = 2$ and $O(n^3)$ for $d = 3$. No better algorithms are known. In contrast, $Q(P)$ and $B(P)$ can be computed in optimal time $O(n)$ for $d = 2, 3$ [3]. This already shows that $C(P)$ is a more complicated object then $B(P)$, and that we should be more modest regarding the quality of approximation we can expect. Actually, $C(P)$ is not even well-defined, because there is not necessarily a unique smallest enclosing cuboid, see Figure 6.2. When we are writing $C(P)$, we therefore mean *some* smallest enclosing cuboid, and with a little care, this poses no problems. For example, the quantity $\text{vol}(C(P))$ is well-defined, because it does not depend on the particular choice of $C(P)$.

Figure 6.2: A set may have more than one smallest enclosing cuboid

Here is the main result of this section.

Theorem 6.1.1 *For $P \subseteq \mathbb{R}^d$ with $|P| = n$, we can compute in $O(d^2 n)$ time a cuboid C that contains P and satisfies*

$$\text{vol}(C) \leq 2^d d! \text{vol}(C(P)).$$

This means, for any constant dimension d, we can approximate the volume of the smallest cuboid that contains P up to some constant factor; however, this factor is already pretty bad when the dimension is only moderately high. On the other hand, the result is not as bad as it might look like, and the exponential dependence on d seems very hard to avoid.

To see this, let's look at smallest enclosing balls again. In Chapter 5, we have shown that we can find an enclosing ball B of P whose radius is at most $(1 + \varepsilon)$ times the radius of $B(P)$, for any $\varepsilon > 0$. With respect to *volume*, this means that

$$\text{vol}(B) \leq (1 + \varepsilon)^d \text{vol}(B(P)),$$

which is exponential in d. In view of this, the bound in Theorem 6.1.1 starts to look somewhat more reasonable.

In order to prove the theorem, we first describe an algorithm for computing some bounding cuboid C and then argue about the quality of C. Actually, the algorithm

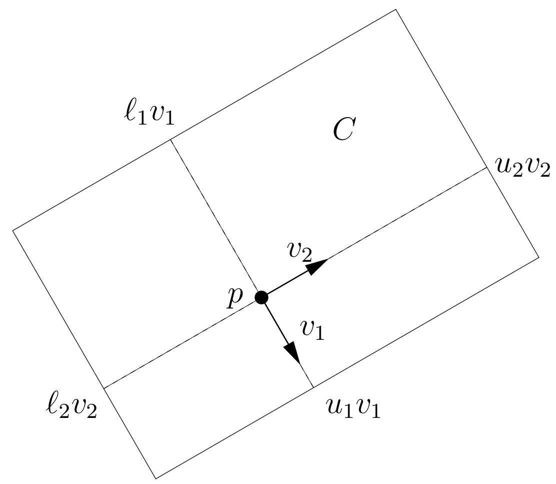

Figure 6.3: Cuboid defined by the algorithm, $d = 2$

computes a point $p \in \mathbb{R}^d$, a set of *pairwise orthogonal*[1] unit vectors $v_1, \ldots, v_d$, and numbers $\ell_k \leq u_k, k = 1, \ldots, d$ such that

$$C = \{p + \sum_{k=1}^{d} \lambda_k v_k \mid \ell_k \leq \lambda_k \leq u_k, k = 1, \ldots, d\}, \tag{6.2}$$

see Figure 6.3. In the exercises, we ask you to verify that this set C is indeed a cuboid.

We will assume that $\mathbf{0} \in P$. This can be achieved through a translation of P in time $O(dn)$. Now we call the following recursive algorithm with parameters (P, d), meaning that the global invariant holds in the beginning. The algorithm constructs v_k, l_k, u_k one after another, starting with $k = d$. The point p used in (6.2) will be $p = 0$.

`MinCuboid_Approx`(P, k):
  (* Global invariant: $p \cdot v_i = 0$ for $p \in P, i = k+1, \ldots, d$ *)
  choose $q \in P$ such that $\|q\|$ is maximum
  `IF` $q \neq \mathbf{0}$ `THEN`
    $v_k = q/\|q\|$
    $\ell_k = \min_{p \in P} p \cdot v_k$
    $u_k = \|q\|$
    $P' = \{p - (p \cdot v_k) v_k \mid p \in P\}$
    (* Local invariant: $p' \cdot v_k = 0$ for $p' \in P'$ *)
    `MinCuboid_Approx`$(P', k-1)$
  `END`

Before we analyze the algorithm, let us try to get a geometric intuition for the top-level call with $k = d$ (Figure 6.4). The vector v_d is a unit vector pointing in the direction

[1]Two vectors v, w are orthogonal if $v \cdot w = 0$.

of some point q farthest away from $\mathbf{0} \in P$. Because $p \cdot v_d$ is the (signed) length of p's projection onto the direction v_d, the value $u_d - \ell_d = \|q\| - \ell_d$ is the extent of P along direction v_d. P' is the projection of P onto the unique hyperplane h through the origin that is orthogonal to v_d. For P', the recursive call finds a $(d-1)$-dimensional bounding cuboid C' within h, which we then extend along direction v_d to a cuboid C containing P.

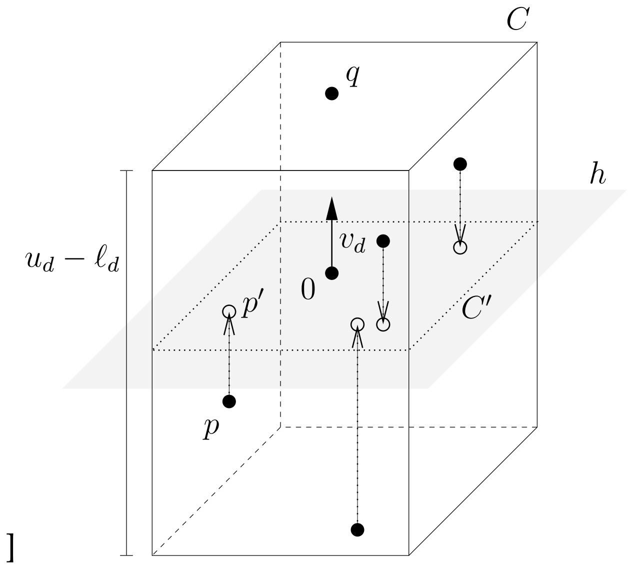

Figure 6.4: Geometric picture of the algorithm

6.1.1 Correctness and runtime of the algorithm.

Because $\|v_k\| = 1$ and $p' \cdot v_k = p \cdot v_k - (p \cdot v_k)\|v_k\|^2 = 0$, the local invariant holds for all $p' \in P'$. The global invariant also holds in the recursive call, because of the just established local invariant, and because the global invariant for $p, q \in P$ shows that for $i = k+1, \ldots, d$,

$$\begin{aligned} p' \cdot v_i &= (p - (p \cdot v_k)v_k) \cdot v_i \\ &= \underbrace{p \cdot v_i}_{=0} - (p \cdot v_k)(v_k \cdot v_i) = -(p \cdot v_k)\underbrace{(q \cdot v_i)}_{=0}/\|q\| = 0. \end{aligned}$$

The latter equation also shows that $v_k \cdot v_i = 0, i = k+1, \ldots, d$ which yields the pairwise orthogonality of all the v_k. Because there are only d pairwise orthogonal nonzero vectors, the recursion bottoms out for some value $\underline{k} \geq 0$. This implies the runtime of $O(d^2 n)$.

To prove that the cuboid

$$C = \{\sum_{i=\underline{k}+1}^{d} \lambda_i v_i \mid \ell_i \leq \lambda_i \leq u_i, i = \underline{k}+1, \ldots, d\} \tag{6.3}$$

contains P, we proceed by induction on d. For $d = 0$, or if the condition of the IF clause fails, we have $P = C = \{\mathbf{0}\}$ and $\underline{k} = d$, so the claim holds. Now assume $d > 0$, and we have already proved the claim for $d - 1$. Inductively, we then know that all points $p' \in P'$ can be written in the form

$$p' = \sum_{i=k_0+1}^{d-1} \lambda_i v_i, \quad \ell_i \le \lambda_i \le u_i, i = \underline{k}+1, \ldots, d-1. \tag{6.4}$$

Furthermore, the definition of p' yields

$$p = p' + (p \cdot v_d)v_d, \tag{6.5}$$

where

$$\ell_d \le p \cdot v_d \le |p \cdot v_d| \le \|p\|\|v_d\| = \|p\| \le \|q\| = u_d,$$

by the Cauchy-Schwarz inequality. Therefore, setting $\lambda_d = (p \cdot v_d)$ and plugging (6.4) into (6.5) gives the desired conclusion $p \in C$.

6.1.2 Quality of the algorithm

It remains to bound the volume of the cuboid C resulting from the algorithm according to (6.3). If the value of $\underline{k}$ for which the recursion bottoms out satisfies $\underline{k} > 0$, we have $\mathrm{vol}(C) = 0$ and are done, so we will assume that $\underline{k} = 0$.

Let $P^{(k)}, k = 1, \ldots, d$, be the iterated projection of P that the algorithm considers in the recursive call with second parameter k. We have

$$P^{(d)} = P, \quad P^{(d-1)} = \{p - (p \cdot v_d)v_d \mid p \in P^{(d)}\},$$

and using the pairwise orthogonality of the v_k, we can inductively verify the general formula

$$P^{(k)} = \{p - \sum_{i=k+1}^{d} (p \cdot v_i)v_i \mid p \in P\}, \quad k = 1, \ldots, d. \tag{6.6}$$

Let $q_k \in P^{(k)}$ be the point chosen for second parameter k, and let $p_k \in P$ be the point of P whose (iterated) projection q_k is. As a consequence of the previous equation we have

$$q_k = p_k - \sum_{i=k+1}^{d} (p_k \cdot v_i)v_i, \quad k = 1, \ldots, d. \tag{6.7}$$

The approximation ratio of Theorem 6.1.1 is derived using two ingredients: a *lower* bound on the volume of the smallest enclosing cuboid $C(P)$, and an *upper* bound on the volume of the cuboid C computed by the algorithm.

A lower bound for $\mathrm{vol}(C(P))$**.** We consider the two sets

$$S_q = \mathrm{conv}(\{\mathbf{0}, q_d, \ldots, q_1\}), \quad S_p = \mathrm{conv}(\{\mathbf{0}, p_d, \ldots, p_1\}).$$

Because the q_i are pairwise orthogonal, they are in particular linearly independent. In this case, we can apply the well-known formula for the volume of a *simplex* in $\mathbb{R}^d$ to obtain

$$\mathrm{vol}(S_q) = \frac{1}{d!}|\det(q_d, \ldots, q_1)|.$$

On the other hand, (6.7) shows that q_k equals p_k plus some linear combination of the $q_i, i > k$ (note that v_i is a multiple of q_i). Recalling that the determinant of a set of vectors does not change when we add to some vector a linear combination of the other vectors, we consequently get (how?) that

$$\det(q_d, \ldots, q_1) = \det(p_d, \ldots, p_1),$$

and therefore

$$\mathrm{vol}(S_p) = \mathrm{vol}(S_q) = \frac{1}{d!}\prod_{k=1}^{d} \|q_k\|,$$

using orthogonality of the q_k. Geometrically, this shows that we can transform the simplex S_q into the simplex S_p, by always moving one of the points *parallel* to its opposite side. During such a movement, the volume stays constant, see Figure 6.5.

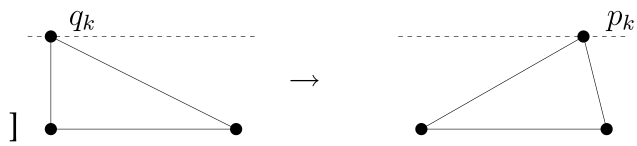

Figure 6.5: Moving a vertex parallel to the opposite side does not change the volume

Now we have a lower bound for the volume of the smallest cuboid $C(P)$. Because $S_p \subseteq \mathrm{conv}(P) \subseteq C(P)$, we get

$$\mathrm{vol}(C(P)) \geq \mathrm{vol}(\mathrm{conv}(P)) \geq \mathrm{vol}(S_p) = \frac{1}{d!}\prod_{k=1}^{d} \|q_k\|. \tag{6.8}$$

An upper bound for $\mathrm{vol}(C)$**.** By construction of C, we have

$$\mathrm{vol}(C) = \prod_{k=1}^{d}(u_k - \ell_k),$$

because $u_k - \ell_k$ is the extent of C in direction of the unit vector v_k. Let $\underline{p}$ be the point defining ℓ_k in the recursive call of the algorithm with second parameter k. We get

$$\begin{aligned} u_k - \ell_k &\leq |u_k| + |\ell_k| = \|q_k\| + |\underline{p} \cdot v_k| \\ &\leq \|q_k\| + \|\underline{p}\|\|v_k\| && \text{(Cauchy-Schwarz inequality)} \\ &= \|q_k\| + \|\underline{p}\| && (\|v_k\| = 1) \\ &\leq 2\|q_k\|. && \text{(choice of } q\text{)} \end{aligned}$$

It follows that

$$\mathrm{vol}(C) \leq 2^d \prod_{k=1}^{d} \|q_k\| \stackrel{(6.8)}{\leq} 2^d d! \mathrm{vol}(C(P)),$$

which is what we wanted to prove in Theorem 6.1.1.

Looking at (6.8), we see that we have actually proved a little more.

Corollary 6.1.2 *Let C be the cuboid computed by a call to* `MinCuboid_Approx`(P, d)*, $P \subseteq \mathbb{R}^d$. Then*

$$\frac{\mathrm{vol}(C(P))}{\mathrm{vol}(\mathrm{conv}(P))} \leq \frac{\mathrm{vol}(C)}{\mathrm{vol}(\mathrm{conv}(P))} \leq 2^d d!.$$

We therefore have shown that $C(P)$ is strictly better than $Q(B)$ and $B(P)$ when it comes to approximating the volume: the volume of any smallest enclosing cuboid of P is only by a *constant factor* (depending on d) larger than the volume of $\mathrm{conv}(P)$. The same factor even holds for the cuboid C that we can easily compute in $O(d^2 n)$ time.

This cuboid satisfies an additional property that we mention without proof. This goes back to a paper by Barequet and Har-Peled [1] that also contains a sketch of the algorithm `MinCuboid_Approx`.

Theorem 6.1.3 *Let C be the cuboid computed by a call to* `MinCuboid_Approx`(P, d)*, $P \subseteq \mathbb{R}^d$. There exists a constant $\alpha_d > 0$ (only depending on d) such that C, scaled with α_d and suitably translated, is contained in $\mathrm{conv}(P)$, see Figure 6.6.*

This means that we will find a not-too-small copy of C inside $\mathrm{conv}(P)$, and this is exploited in a number of approximation algorithms for other problems.

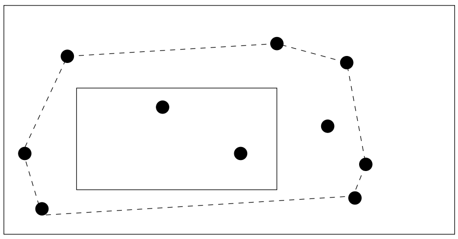

Figure 6.6: Bounding and *inscribed* cuboid of the same shape

The proof of Theorem 6.1.3 (and similar statements) involves properties of an even nicer family of bounding volumes that we introduce next.

6.2 Ellipsoids

Definition 6.2.1

(i) An ellipsoid *in* $\mathbb{R}^d$ *is any set of the form*

$$E = \{x \in \mathbb{R}^d \mid (x-c)^T A(x-c) \leq z,$$

where $c \in \mathbb{R}^d$ *is the* center *of the ellipsoid,* A *is a positive definite matrix,*[2] *and* $z \in \mathbb{R}$.

(ii) For $\lambda \geq 0$*, the* scaled *ellipsoid* λE *is the set*

$$\lambda E = \{c + \lambda(x-c) \mid x \in E\}.$$

This scales E *relative to its center, and it is easy to prove (Exercises) that* λE *is an ellipsoid again.*

If A is the identity matrix, for example, E is a ball with center c and squared radius z. In general, any ellipsoid is the affine image of a ball, and the image of any ellipsoid under a nonsingular affine transformation is again an ellipsoid.

Here is what makes ellipsoids attractive as bounding (and also as inscribed) volumes. This is classic material [2].

Theorem 6.2.2 *Let* $K \subseteq \mathbb{R}^d$ *be a convex body (this is a convex and compact set of positive volume).*

(i) There is a unique ellipsoid $\overline{E}(K)$ *of smallest volume such that* $K \subseteq \overline{E}(K)$ *and a unique ellipsoid* $\underline{E}(K)$ *of largest volume such that* $\underline{E}(K) \subseteq K$.

(ii) $1/d\, \overline{E}(K) \subseteq K$

(iii) $d\underline{E}(K) \supseteq K$.

(iv) If K *is* centrally symmetric *(*$p \in K \Rightarrow -p \in K$*), we have* $1/\sqrt{d}\, \overline{E}(K) \subseteq K$ *and* $\sqrt{d}\underline{E}(K) \supseteq K$.

This is a theorem, similar in spirit to Theorem 6.1.3. The difference is that the scaling factors are explicit (and small), and that no additional translation is required. The theorem says that any convex body is 'wedged' between two concentric ellipsoids whose scale only differs by a factor of d at most.

[2] $x^T Ax > 0$ for $x \neq 0$.

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