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Approximation Algorithms and Semidefinite Programming FS12
Exercise Set 6

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Discussion: Mai 22 & 29

Exercise 1 (Vector discrepancy)

[Exercise 11.1]

a) Show that every set system on n points has vector discrepancy at most $\sqrt{n}$.

b) Show that this bound is tight, possibly up to a multiplicative constant independent of n.

c) Let $\mathcal{F}$ be a system on n sets on a set V of n points, such that every point is contained in exactly r sets of $\mathcal{F}$, and for every two distinct points $i, j \in V$, there are exactly t sets $F \in \mathcal{F}$, with $\{i, j\} \subset F$. Prove that $\text{vecdisc}(\mathcal{F}) \geq \sqrt{r - t}$.

Hint - Exercise 1.c) Instead of $\max_{F\in\mathcal{F}} \| \sum_{j\in F} \mathbf{u}_j \|^2$, estimate $\sum_{F\in\mathcal{F}} \| \sum_{j\in F} \mathbf{u}_j \|^2$.

Exercise 2 (Discrepancy I)

[Exercise 11.2] Use the probabilistic method to show that $\text{disc}(\mathcal{F}) = O(\sqrt{n \log m})$ for every system of m sets on n points.

Hint - Chernoff bound: Let $X_1, \ldots X_n$ be independent random variables with $0 \leq X_i \leq 1$, for $i = 1, \ldots, n$ and let $X = X_1 + \cdots + X_n$, $\mu = \mathbb{E}[X]$. Then for all $0 \leq \epsilon \leq 1$: $\mathbf{Pr}[|X - \mu| \geq \epsilon\mu] \leq 2\exp\left(-\frac{\epsilon^2}{3}\mu\right)$.

Exercise 3* (Discrepancy II) [*Bonus exercise]

[Exercise 11.3] Use the probabilistic method to show the existence of set systems with n^2 sets on n points and with discrepancy $\Omega(\sqrt{n \log n})$.

(Together with Exercise 1.a) this shows that the gap between vecdisc and disc can be at least of order $\sqrt{\log m}$, for $m = n^2$. The complete set system 2^V exhibits a similar gap for $m = 2^n$.)

Hint: Fix an arbitrary coloring χ and show that the discrepancy of χ for a system of n^2 independent random sets is below $c\sqrt{n \log n}$ with probability smaller than 2^{-n}. A random set is obtained by including each point independently with probability $\frac{1}{2}$.