Exercise 1 (Rank-constrained programs) (5 Points)

[Exercise 2.4] A rank-constrained semidefinite program is a problem of the form

$$\begin{align*}
\max \quad & C \cdot X \\
\text{s.t.} \quad & A(X) = b \\
& X \succeq 0 \\
& \text{Rank}(X) = k,
\end{align*}$$

where $k$ is a fixed integer. Show that the problem of solving a rank-constrained semidefinite program is NP-hard for $k = 1$.

Exercise 2 (Approximating Max-2-Sat) (10 Points)

Let $X$ be a set of variables. A literal is any variable $x \in X$ or its negation $\bar{x}$. A $k$-clause is a disjunction of $k$ literals and a $k$-CNF formula is a conjunction of $k$-clauses. A clause is satisfied by an assignment $f : X \mapsto \{0, 1\}$ if it evaluates to 1 (true) under this assignment.

Max-2-Sat is the following computational problem: Given a 2-CNF formula in $n$ variables, determine the maximal number of clauses that can be simultaneously satisfied.

a) Show that we get an $3/4$-approximation algorithm by assigning each variable to 0 or 1 with probability $p = 1/2$ each, independent of the other variables.

b) Find a polynomial 0.878-approximation algorithm for Max-2-Sat.

Exercise 3 (Optimization over polynomials) (10 Points)

[Exercise 2.7] Let $p(x) \in \mathbb{R}[x]$ be a univariate polynomial of degree $d$ with real coefficients. We would like to decide whether $p(x)$ is a sum of squares, i.e., if it can be written as $p(x) = q_1(x)^2 + \cdots + q_m(x)^2$ for some $q_1(x), \ldots, q_m(x) \in \mathbb{R}[x]$. Formulate this problem as the feasibility of a semidefinite program.

b) Let us call a polynomial $p(x) \in \mathbb{R}[x]$ nonnegative if $p(x) \geq 0$ for all $x \in \mathbb{R}$. Obviously, a sum of squares is nonnegative. Prove that the converse holds as well.

c) Let $p(x) \in \mathbb{R}[x]$ be a given polynomial. Express its global minimum $\min\{p(x) \mid x \in \mathbb{R}\}$ as the optimum of a suitable semidefinite program.
**Hint - Exercise 2:** Proceed similarly as in the MaxCut approximation. Formulate the problem as an equivalent integer quadratic program. Introduce a variable $y_i$ for each variable $x_i$ of the Max-2-SAT problem and introduce an additional variable $y_0$ to encode true and false.

**Hint - Exercise 3.a)** Find suitable $z$ and $A$, s.t. $p(x) = z^T Az$.

**Hint - Exercise 3.b)** Factor into quadratic polynomials.