Exercise 1 (Sum of the $k$ largest eigenvalues) (10 Points)

[Exercise 4.11] Let $C \in \text{SYM}_n$.

a) Prove that the value of the following cone program is the sum of the $k$ largest eigenvalues of $C$.

\[
\begin{align*}
\text{min} \quad & ky + \text{Tr}(Y) \\
\text{s.t.} \quad & gI_n + Y - C \succeq 0 \\
& (Y, y) \in \text{PSD}_n \oplus \mathbb{R}.
\end{align*}
\]

b) Derive the dual program (see Section 4.7) and show that its value is also the sum of the $k$ largest eigenvalues of $C$.

Exercise 2 (A Semidefinite Program for the Theta Function) (10 Points)

[Exercise 4.12] For a given graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$, consider the semidefinite program

\[
\begin{align*}
\text{maximize} \quad & J_n \cdot X \\
\text{subject to} \quad & \text{Tr}(X) = 1 \\
& x_{ij} = 0, \quad \{i, j\} \in E \\
& X \succeq 0,
\end{align*}
\]

where $J_n$ is the all-one $n \times n$ matrix, and show that its value is $\vartheta(G)$.

Exercise 3 (Theta Function of the strong product) (5 Points)

[Exercise 4.12] Prove that Lemma 3.4.2 actually holds with equality. You have to show

\[
\vartheta(G \cdot H) = \vartheta(G) \vartheta(H)
\]

for all graphs $G, H$. 
**Hint - Exercise 1.a)** You may use the statement of Exercise 4.10.

**Hint - Exercise 2)** Dualize the semidefinite program in Theorem 3.6.1.

**Hint - Exercise 3)** Use the expression of $\vartheta(G)$ from Exercise 2.