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Approximation Algorithms and Semidefinite Programming *FS12*
Homework 3

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Due date: Mai 11, 2012, 10h15.

- Solutions are to be handed in typed in LaTeX.
- Please bring a print-out of your solution with you to the lecture. If you cannot attend, you may alternatively send your solution as a PDF to `stich@inf.ethz.ch`.
- Proofs are to be formally correct, complete and clearly explained. Be short and precise!

Exercise 1 (Optimal solutions for MaxCut SDP) *(12.5 Points)*

[Exercises 8.1 & 8.2] Let $G = (\{1, \ldots, n\}, E)$ be a graph on n vertices. Recall that the Goemans-Williamson semidefinite relaxation of MaxCut, written as a vector program, is:

$$\begin{array}{rll} \mathsf{SDP} = \text{Maximize} & \sum_{\{i,j\}\in E} \frac{1-\mathbf{v}_i^T\mathbf{v}_j}{2} & \\ \text{subject to} & \|\mathbf{v}_i\| = 1\,, & i = 1, \ldots, n\,. \end{array} \tag{GW}$$

Any solution of this SDP determines a vector representation $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S^{n-1}$ of the vertices of G.

a) Let G be a bipartite graph. What is the optimum value of the vector program (GW), and what does the corresponding vector representation look like?

b) Prove that the five vectors arranged in a regular pentagon, as shown in the picture in 8.3.3, constitute an *optimal* solution of the vector program (GW) for the 5-cycle C_5.

Exercise 2 (Euclidean distance matrices) *(12.5 Points)*

[Exercises 2.5 & 2.6] A Matrix $M \in \mathbb{R}^{n\times n}$ is called a *Euclidean distance matrix* if there exists n points $\mathbf{p}_1, \ldots \mathbf{p}_n \in \mathbb{R}^n$, such that

$$m_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|^2\,, \qquad 1 \le i, j \le m\,.$$

a) Prove that a matrix M is a Euclidean distance matrix if and only if M is symmetric, $m_{ii} = 0$ for $i = 1, \ldots, n$ and

$$\mathbf{x}^T M\mathbf{x} \le 0\,, \quad \text{for all } \mathbf{x} \text{ with } \sum_{i=1}^{n} x_i = 0\,. \tag{1}$$

b) Let $G = (\{1, \ldots, n\}, E)$ be a graph with two edge weight functions $\alpha, \beta\colon E \to \mathbb{R}$ with $\alpha(e) \le \beta(e)$, $e \in E$. We want to know whether there exists points $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^n$, such that

$$\alpha(\{i,j\}) \le \|\mathbf{p}_i - \mathbf{p}_j\|^2 \le \beta(\{i,j\})\,, \qquad \text{for all } \{i,j\} \in E\,. \tag{2}$$

Show that this decision problem can be formulated as a semidefinite program.