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Approximation Algorithms and Semidefinite Programming FS12
Homework 4

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Due date: Mai 25, 2012, 10h15.

- Solutions are to be handed in typed in LaTeX.
- Please bring a print-out of your solution with you to the lecture. If you cannot attend, you may alternatively send your solution as a PDF to `stich@inf.ethz.ch`.
- Proofs are to be formally correct, complete and clearly explained. Be short and precise!

Exercise 1 (Maximizing quadratic forms) *(12.5 Points)*

[Exercise 10.3] Let $A \in R^{n\times n}$ be a positive definite matrix and consider the optimization problem

$$\mathsf{OPT} := \max\left\{\mathbf{x}^T A\mathbf{x} \colon \mathbf{x} \in \{-1,1\}^n\right\}. \tag{1}$$

We relax this problem to the vector program

$$\mathsf{SDP} := \max\left\{\sum_{i,j} a_{ij}\mathbf{v}_i^T\mathbf{v}_j \colon \|\mathbf{v}_i\| = 1, i = 1,\dots,n\right\}. \tag{2}$$

Let $\mathbf{v}_1^*,\dots,\mathbf{v}_n^*$ be an optimal solution of (2). Then a feasible solution $\mathbf{y} \in R^n$ to (1) can be obtained from the $\mathbf{v}_i^*$'s by the random hyperplane rounding (again as in Goemans-Williamson). The goal of this exercise is to show that the expected approximation ratio of this algorithm is at least $\frac{2}{\pi} \approx 0.636619\dots$.

a) Let $X \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix, $|x_{ij}| \leq 1$ for all $1 \leq i,j \leq n$ and let us define $\arcsin[X] := (\arcsin x_{ij})$. Show that $\arcsin[X] - X \succeq 0$.

b) Starting from an optimal solution $\mathbf{v}_1^*,\dots,\mathbf{v}_n^*$ of (2), calculate the expected value of the rounded solution $\mathbf{y}$ and conclude that $\mathsf{OPT} \geq \frac{2}{\pi}\mathsf{SDP}$.

c) Let A as above. Show that the value OPT of the optimization problem (1) equals

$$\mathsf{OPT}' = \max\left\{\frac{2}{\pi}A \bullet \arcsin[X] \colon X \succeq 0, X_{ii} = 1, i = 1,\dots,n\right\}. \tag{3}$$

Hint - Exercise 1.a) Consider the Taylor series of $\arcsin x - x$ and use the following fact (which you don't need to prove):

For matrices $X, Y \in R^{n\times n}$, define the entrywise – or *Hadamard* – product $X \circ Y := (x_{ij}y_{ij})$.
Schur's Theorem: If $X \succeq 0$ and $Y \succeq 0$, then $X \circ Y \succeq 0$.

Exercise 2 (Coloring 3-Colorable Graphs) *(12.5 Points)*

In this exercise we will derive a coloring algorithm that uses $\tilde{O}(n^{0.387})$ colors to color 3-colorable graphs. Note that this algorithms performs worse than the KMS-scheme from the lecture. We use semidefinite programming to first obtain a *semicoloring* of the graph. A semicoloring is a coloring of the nodes of a n vertex graph G such that at most $n/4$ edges have endpoints of the same color (i.e. at least $n/2$ vertices are colored such that any edge between them has endpoints that are colored differently).

Let $G = (V, E)$ be a 3-colorable graph on n vertices, m edges and with maximal degree Δ. Consider the following vector program, where we have a vector $\mathbf{v}_i$ for every vertex $i \in V$:

$$\mathsf{SDP} := \min \left\{ t \colon \mathbf{v}_i^T \mathbf{v}_j \le t \,, \forall \{i,j\} \in E \,, \|\mathbf{v}_i\| = 1 \,, i = 1, \dots, n \right\} . \tag{4}$$

We already know (cf. Section 3.7) that for every 3-colorable graph G there is a feasible solution to (4) with value $\mathsf{SDP} \le -\frac{1}{2}$. Let $\mathbf{v}_1^*, \dots, \mathbf{v}_n^*$ be an optimal solution of (2). We can obtain a (semi)coloring by the following rounding procedure: Chose $r = 2 + \log_3 \Delta$ random unit vectors $\mathbf{r}_1, \dots, \mathbf{r}_r$. The r vectors define 2^r different regions into which the vectors $\mathbf{v}_i^*$ can fall: one region for each possibility of whether $\mathbf{r}_i^T \mathbf{v}_j^* \ge 0$ or $\mathbf{r}_i^T \mathbf{v}_j^* < 0$ for all $i = 1, \dots, r$. We then color the vectors in each region with a distinct color.

a) Suppose you have an algorithm that produces a semicoloring for G with k colors. Show how we can obtain a valid coloring of G using $O(k \log n)$ colors.

b) Prove that for the above rounding scheme and every edge $e = \{i, j\}$ of G:

$$\mathbf{Pr}\left[i \text{ and } j \text{ get the same color} \right] \le \frac{1}{9\Delta} .$$

c) Conclude that the rounding scheme produces a semicoloring of G with $4\Delta^{\log_3 2} \approx 4\Delta^{0.613}$ colors with probability at least $1/2$.

d) Use Wigderson's trick to obtain an algorithm that semicolors every 3-colorable graph with $O(n^{\log_6 2})$ colors with probability at least $1/2$.