Exercise 1 (Maximizing quadratic forms) \((12.5\text{ Points})\)

[Exercise 10.3] Let \(A \in \mathbb{R}^{n \times n}\) be a positive definite matrix and consider the optimization problem

\[
\text{OPT} := \max \left\{ x^T A x : x \in \{-1, 1\}^n \right\}.
\]  

We relax this problem to the vector program

\[
\text{SDP} := \max \left\{ \sum_{i,j} a_{ij} v_i^T v_j : \|v_i\| = 1, i = 1, \ldots, n \right\}.
\]

Let \(v_1^*, \ldots, v_n^*\) be an optimal solution of \((2)\). Then a feasible solution \(y \in \mathbb{R}^n\) to \((1)\) can be obtained from the \(v_i^*\)'s by the random hyperplane rounding (again as in Goemans-Williamson). The goal of this exercise is to show that the expected approximation ratio of this algorithm is at least \(2 \pi \approx 0.636619\ldots\).

\[\text{a)}\ \text{Let } X \in \mathbb{R}^{n \times n}\text{ be a positive semidefinite matrix, } |x_{ij}| \leq 1 \text{ for all } 1 \leq i, j \leq n \text{ and let us define } \arcsin[X] := (\arcsin x_{ij}). \text{ Show that } \arcsin[X] - X \succeq 0.\]

\[\text{b)}\ \text{Starting from an optimal solution } v_1^*, \ldots, v_n^* \text{ of } \text{SDP}, \text{ calculate the expected value of the rounded solution } y \text{ and conclude that } \text{OPT} \geq \frac{2}{\pi} \text{SDP}.\]

\[\text{c)}\ \text{Let } A \text{ as above. Show that the value } \text{OPT} \text{ of the optimization problem } \text{(1)} \text{ equals}\]

\[
\text{OPT}' = \max \left\{ \frac{2}{\pi} A \bullet \arcsin[X] : X \succeq 0, X_{ii} = 1, i = 1, \ldots, n \right\}.
\]

**Hint - Exercise 1.a)** Consider the Taylor series of \(\arcsin x - x\) and use the following fact (which you don’t need to prove):

For matrices \(X, Y \in \mathbb{R}^{n \times n}\), define the entrywise – or Hadamard – product \(X \circ Y := (x_{ij}y_{ij})\).

**Schur’s Theorem:** If \(X \succeq 0\) and \(Y \succeq 0\), then \(X \circ Y \succeq 0\).
Exercise 2 (Coloring 3-Colorable Graphs) \( (12.5 \text{ Points}) \)

In this exercise we will derive a coloring algorithm that uses \( \tilde{O}(n^{0.387}) \) colors to color 3-colorable graphs. Note that this algorithm performs worse than the KMS-scheme from the lecture. We use semidefinite programming to first obtain a semicoloring of the graph. A semicoloring is a coloring of the nodes of a \( n \) vertex graph \( G \) such that at most \( n/4 \) edges have endpoints of the same color (i.e. at least \( n/2 \) vertices are colored such that any edge between them has endpoints that are colored differently).

Let \( G = (V, E) \) be a 3-colorable graph on \( n \) vertices, \( m \) edges and with maximal degree \( \Delta \). Consider the following vector program, where we have a vector \( v_i \) for every vertex \( i \in V \):

\[
\text{SDP} := \min \left\{ t : v_i^T v_j \leq t, \forall \{i, j\} \in E, \|v_i\| = 1, i = 1, \ldots, n \right\}.
\]

(4)

We already know (cf. Section 3.7) that for every 3-colorable graph \( G \) there is a feasible solution to (4) with value \( \text{SDP} \leq -\frac{1}{2} \). Let \( v_1^*, \ldots, v_n^* \) be an optimal solution of (4). We can obtain a (semi)coloring by the following rounding procedure: Chose \( r = 2 + \log_3 \Delta \) random unit vectors \( r_1, \ldots, r_r \). The \( r \) vectors define \( 2^r \) different regions into which the vectors \( v_i^* \) can fall: one region for each possibility of whether \( r_i^T v_j^* \geq 0 \) or \( r_i^T v_j^* < 0 \) for all \( i = 1, \ldots, r \). We then color the vectors in each region with a distinct color.

a) Suppose you have an algorithm that produces a semicoloring for \( G \) with \( k \) colors. Show how we can obtain a valid coloring of \( G \) using \( O(k \log n) \) colors.

b) Prove that for the above rounding scheme and every edge \( e = \{i, j\} \) of \( G \):

\[
\Pr \left[ i \text{ and } j \text{ get the same color} \right] \leq \frac{1}{9\Delta}.
\]

c) Conclude that the rounding scheme produces a semicoloring of \( G \) with \( 4\Delta^{\log_3 2} \approx 4\Delta^{0.613} \) colors with probability at least \( 1/2 \).

d) Use Wigderson’s trick to obtain an algorithm that semicolors every 3-colorable graph with \( O(n^{0.65}) \) colors with probability at least \( 1/2 \).