4. Delaunay Triangulations

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4.1 Point Set Triangulations Revisited

In the previous chapter, we have defined triangulations (straight-line embeddings of graphs where all bounded faces are triangles). In Figure 3.3, we have depicted three classes of triangulations: point set triangulations, simple polygon triangulations, and convex polygon triangulations. This chapter is about point set triangulations, and in particular about Delaunay triangulations that are specific point set triangulations.

**Definition 4.1** Given a finite point set $P \subseteq \mathbb{R}^2$. A triangulation of $P$ is a triangulation whose vertices are exactly the points of $P$, and whose outer face is the convex hull of $P$.

While it was clear that one can always find a triangulation of a convex polygon (we were even able to count the number of possibilities), this is maybe less clear but still easy for point sets.

**Observation 4.2** Given a finite point set $P \subseteq \mathbb{R}^2$ with the property that not all points of $P$ are on a common line. Then $P$ has a triangulation.

To construct one, we assume that no two points of $P$ have the same x-coordinate (we can always rotate to achieve this). Let $p_1, \ldots, p_n$ be the list of points, ordered by x-coordinate. Let $p_1, \ldots, p_m$ be the smallest prefix such that $p_1, \ldots, p_m$ are not on a common line. We triangulate $p_1, \ldots, p_m$ by connecting $p_m$ to $p_1, \ldots, p_{m-1}$ (which are on a common line), see Figure 4.1 (left).

![Figure 4.1: Constructing the scan triangulation of P](image)

Then we add $p_{m+1}, \ldots, p_n$. In adding $p_i, i > m$, we connect $p_i$ with all vertices of $\text{conv}\{p_1, \ldots, p_{i-1}\}$ that it "sees". Since there are always at least two such vertices, we always add triangles (Figure 4.1 right). When we are done, exactly the points from $P$ show up as vertices of the triangulation.
The triangulation that we get in this way is called a \textit{scan triangulation}. Such a triangulation (Figure 4.2 (left) shows a larger one) is usually “ugly”, though, since it tends to have many long and skinny triangles. In contrast, the \textit{Delaunay triangulation} of the same point set (Figure 4.2 (right)) looks much nicer, and we discuss next how to get this triangulation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{triangulations.png}
\caption{Two triangulations of the same 50-point set}
\end{figure}

4.2 The Empty Circle Property

The \textit{circumcircle} of a triangle is the unique circle passing through the three vertices of the triangle, see Figure 4.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{circumcircle.png}
\caption{Circumcircle of a triangle}
\end{figure}

\textbf{Definition 4.3} \textit{Given a finite point set }$P \subseteq \mathbb{R}^2$. A triangulation of $P$ is called \text{Delaunay triangulation} if every triangle has an \text{empty circumcircle}, meaning that the \text{interior of the circle} does not contain any point of $P$. 
Figure 4.4 illustrates this: it shows a Delaunay triangulation of a set of 6 points: the circumcircles of all five triangles are empty (we also say that the triangles satisfy the empty circle property). The dashed circle is not empty, but that’s ok, since it is not a circumcircle of any triangle.

![Delaunay triangulation diagram](image)

**Figure 4.4: All triangles satisfy the empty circle property**

It is instructive to look at the case of four points in convex position. We already know that there are two possible triangulations, but in general, only one of them will be Delaunay, see Figure 4.5 (a) and (b). If all four points are on a common circle, though, this circle is empty; at the same time it is the circumcircle of all possible triangles; therefore, both triangulations of the point set are Delaunay, see Figure 4.5 (c).

![Triangulations of 4-point sets](image)

**Figure 4.5: Triangulations of 4-point sets**
4.3 The Lawson Flip algorithm

It is not clear yet that every point set actually has a Delaunay triangulation (given that not all points are on a common line). In this and the next two sections, we will prove that this is the case. The proof is algorithmic. Here is the Lawson flip algorithm for a set $P$ of $n$ points.

1. Compute some triangulation of $P$ (for example, the scan triangulation)

2. While there exists a subtriangulation of four points in convex position that is not Delaunay (like in Figure 4.5 (b)), replace this subtriangulation by the other triangulation of the four points (Figure 4.5 (a)).

We call the replacement operation in Step 2 a (Lawson) flip.

**Theorem 4.4** Let $P \subseteq \mathbb{R}^2$ be a set of $n$ points, equipped with some triangulation $\mathcal{T}$. The Lawson flip algorithm terminates after at most $\binom{n}{2} = O(n^2)$ flips, and the resulting triangulation $D$ is a Delaunay triangulation of $P$.

4.4 Termination of the Lawson Flip Algorithm: The Lifting Map

In order to prove Theorem 4.4, we invoke the lifting map. This is the following: given a point $p = (x, y) \in \mathbb{R}^2$, its lifting $\ell(p)$ is the point

$$\ell(p) = (x, y, x^2 + y^2) \in \mathbb{R}^3.$$ 

Geometrically, $\ell$ “lifts” the point vertically up until it lies on the unit paraboloid $\{(x, y, z) \mid z = x^2 + y^2\} \subseteq \mathbb{R}^3$, see Figure 4.6 (a).

![The lifting map](image)

**Figure 4.6:** The lifting map: circles map to planes

Here is the important property of the lifting map that is illustrated in Figure 4.6 (b) (proof left as an exercise).
Lemma 4.5 Let \( C \subseteq \mathbb{R}^2 \) be a circle of positive radius. The “lifted circle” \( \ell(C) = \{ \ell(p) \mid p \in C \} \) is contained in a unique plane \( h_C \subseteq \mathbb{R}^3 \). Moreover, a point \( p \in \mathbb{R}^2 \) is strictly inside (outside, respectively) of \( C \) if and only if the lifted point \( \ell(p) \) is strictly below (above, respectively) \( h_C \).

Using the lifting map, we can now prove Theorem 4.4. Let us fix the point set \( P \) for this and the next section. First, we need to argue that the algorithm indeed terminates (if you think about it a little, this is not obvious). So let’s interpret a flip operation in the lifted picture. The flip involves four points in convex position in \( \mathbb{R}^2 \), and their lifted images form a tetrahedron in \( \mathbb{R}^3 \) (think about why this tetrahedron cannot be “flat”).

The tetrahedron is made up of four triangles; when you look at it from the top, you see two of the triangles, and when you look from the bottom, you see the other two. In fact, what you see from the top and the bottom are the lifted images of the two possible triangulations of the four-point set in \( \mathbb{R}^2 \) that is involved in the flip.

Here is the crucial fact that follows from Lemma 4.5: The two top triangles come from the non-Delaunay triangulation before the flip, see Figure 4.7 (a). The reason is that both top triangles have the respective fourth point below them, meaning that in \( \mathbb{R}^2 \), the circumcircles of these triangles contain the respective fourth point—the empty circle property is violated. In contrast, the two bottom triangles come from the Delaunay triangulation of the four points: they both have the respective fourth point above them, meaning that in \( \mathbb{R}^3 \), the circumcircles of the triangles do not contain the respective fourth point, see Figure 4.7 (b).

![Figure 4.7: Lawson flip: the height of the surface of lifted triangles decreases](image)

In the lifted picture, a Lawson flip can therefore be interpreted as an operation that replaces the top two triangles of a tetrahedron by the two bottom ones. If we consider the lifted image of the current triangulation, we therefore have a surface in \( \mathbb{R}^3 \) whose pointwise height can only decrease through Lawson flips. In particular, once an edge has been flipped, this edge will be strictly above the resulting surface and can therefore
never be flipped a second time. Since $n$ points can span at most $\binom{n}{2}$ edges, the bound on the number of flips follows.

### 4.5 Correctness of the Lawson Flip Algorithm

It remains to show that the triangulation of $P$ that we get upon termination of the Lawson flip algorithm is indeed a Delaunay triangulation. Here is a first observation telling us that the triangulation is “locally Delaunay”.

**Observation 4.6** Let $\Delta, \Delta'$ be two adjacent triangles in the triangulation $\mathcal{D}$ that results from the Lawson flip algorithm. Then the circumcircle of $\Delta$ does not have any vertex of $\Delta'$ in its interior, and vice versa.

If the two triangles together form a convex quadrilateral, this follows from the fact that the Lawson flip algorithm did not flip the common edge of $\Delta$ and $\Delta'$. If the four vertices are not in convex position, this is basic geometry: given a triangle $\Delta$, its circumcircle $C$ can only contains points of $C \setminus \Delta$ that form a convex quadrilateral with the vertices of $\Delta$.

Now we show that the triangulation is also “globally Delaunay”.

**Theorem 4.7** The triangulation $\mathcal{D}$ that results from the Lawson flip algorithm is a Delaunay triangulation of $P$.

**Proof.** Suppose for a contradiction that there is some triangle $\Delta \in \mathcal{D}$ and some point $p \in P$ strictly inside the circumcircle $C$ of $\Delta$. Among all such pairs $(\Delta, p)$, we choose one for which we the distance of $p$ to $\Delta$ is minimal. Note that this distance is positive since $\mathcal{D}$ is a triangulation of $P$. The situation is as in Figure 4.8 (a).

![Figure 4.8: Correctness of the Lawson flip algorithm](image-url)

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Now consider the edge \(e\) of \(\Delta\) that is closest to \(p\). There must be another triangle \(\Delta'\) containing the edge \(e\), and by the local Delaunay property of \(\mathcal{D}\), the third vertex \(q\) of \(\Delta'\) is on or outside of \(C\), see Figure 4.8 (b). But then the circumcircle \(C'\) of \(\Delta'\) contains the whole portion of \(C\) on \(p\)'s side of \(e\), hence it also contains \(p\); moreover, \(p\) is closer to \(\Delta'\) than to \(\Delta\) (Figure 4.8 (c)). But this is a contradiction to our choice of \(\Delta\) and \(p\). Hence there was no \((\Delta, p)\), and \(\mathcal{D}\) is a Delaunay triangulation. \(\square\)

### 4.6 The Delaunay Graph

Despite the fact that a point set may have more than one Delaunay triangulation, there are certain edges that are present in every Delaunay triangulation.

**Definition 4.8** The Delaunay graph of \(P \subseteq \mathbb{R}^2\) consists of all line segments \(\overline{pq}, p, q \in P\) that are contained in every Delaunay triangulation of \(P\).

The following characterizes the edges of the Delaunay graph.

**Lemma 4.9** Let \(p, q \in P\). The segment \(\overline{pq}\) is in the Delaunay graph of \(P\) if and only if there exists a circle \(C\) that has \(p\) and \(q\) on the boundary and all other points of \(P\) strictly outside.

**Proof.** Let \(\mathcal{D}\) be a Delaunay triangulation of \(P\). If \(\overline{pq}\) is not an edge of \(\mathcal{D}\), there must be another edge \(\overline{rs}\) of \(\mathcal{D}\) that crosses \(\overline{pq}\) (otherwise, we could add \(\overline{pq}\) to \(\mathcal{D}\) and still have a planar graph, a contradiction to \(\mathcal{D}\) being a triangulation of \(P\)). Moreover, \(r\) and \(s\) are both strictly outside of \(C\). With a similar argument as in the proof of Theorem 4.7 (transforming \(C\) while keeping \(p, q\) on its boundary), we can conclude that both the circumcircles of \(p, q, r\) and of \(p, q, s\) are empty, meaning that \(\{p, q, r, s\}\) has a Delaunay triangulation with diagonal \(\overline{pq}\), see Figure 4.9.

The lifting map interpretation on the other hand implies that the other diagonal \(\overline{rs}\) won’t lead to a Delaunay triangulation of \(\{p, q, r, s\}\). This in turn implies that every circle \(C'\) with \(r, s\) on the boundary contains \(p\) or \(q\) in its interior. This is a contradiction to \(\overline{rs}\) being an edge of a triangle in \(\mathcal{D}\), so \(\overline{pq} \notin \mathcal{D}\) after all. \(\square\)

The Delaunay graph is useful to prove uniqueness of the Delaunay triangulation in case of general position.

**Theorem 4.10** Let \(P\) be in general position, meaning that no 4 points of \(P\) are on a common circle. Then \(P\) has a unique Delaunay triangulation.

**Proof.** Let \(\mathcal{D}\) be some Delaunay triangulation. We prove that the Delaunay graph is equal to \(\mathcal{D}\) which settles the theorem. To this end, let \(\overline{pq}\) be any edge of \(\mathcal{D}\). Since \(\overline{pq}\) is the edge of some Delaunay triangle, there exists an empty circle \(C\) with \(p, q\) and a third point \(r\) on the boundary. We now slightly transform \(C\) into a circle \(C'\) so that \(C'\) still goes through \(p\) and \(q\) but no longer contains \(r\). By general position, we can do this in
4.7 Every Delaunay Triangulation Maximizes the Smallest Angle

Why are we interested in Delaunay triangulations at all? After all, having empty circumcircles is not a goal in itself. But it turns out that Delaunay triangulations satisfy a number of interesting properties. Here we just show one of them.

Given a triangulation \( T \) of \( P \), we consider the sorted sequence \( A(T) = (\alpha_1, \alpha_2, \ldots, \alpha_{3m}) \) of interior angles, where \( m \) is the number of triangles (we have already remarked earlier that \( m \) is a function of \( P \) only and does not depend on \( T \). Being sorted means that \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{3m} \). Let \( T, T' \) be two triangulations. We say that \( A(T) < A(T') \) if there exists some \( i \) for which \( \alpha_i < \alpha'_i \) and \( \alpha_j = \alpha'_j, j < i \). This means that \( A(T) \) is lexicographically smaller than \( A(T') \). Here is the result.

**Theorem 4.11** Let \( P \subseteq \mathbb{R}^2 \) be a finite set of points in general position (not all on a line, and no 4 on a common circle). Let \( D^* \) be the unique Delaunay triangulation of \( P \), and let \( T \) be any triangulation of \( P \). Then

\[
A(T) \leq A(D^*).
\]

In particular, \( D^* \) maximizes the smallest angle among all triangulations of \( P \).
Figure 4.10: Proving angle-optimality of a Delaunay triangulation

Proof.

We know that $T$ can be transformed into $D'$ through the Lawson flip algorithm, and we are done if we can show that each such flip lexicographically increases the sorted angle sequence. A flip replaces 6 interior angles by six other interior angles, and we will actually show that the smallest of the six angles strictly increases under the flip. This implies that the whole angle sequence increases lexicographically.

Let us first look at the situation of four cocircular points, see Figure 4.10 (a). In this situation, the inscribed angle theorem (a generalization of Thales’ Theorem) tells us that the 8 depicted angles come in 4 equal pairs. In part (b) of Figure 4.10, we have the situation in which we perform a Lawson flip (replacing the solid with the dashed diagonal). Here, the notations "$\alpha$ (\overline{\alpha}, respectively)" stands for an angle strictly smaller (larger, respectively) than $\alpha$.

Here are the 6 angles before the flip:

$\alpha_1 + \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \overline{\alpha_1}, \quad \overline{\alpha_2}, \quad \overline{\alpha_3} + \overline{\alpha_4}$.

After the flip, we have

$\alpha_1, \quad \alpha_2, \quad \overline{\alpha_3}, \quad \overline{\alpha_4}, \quad \alpha_1 + \alpha_4, \quad \alpha_2 + \alpha_3$.

Now, for every angle after the flip there is at least one smaller angle before the flip:

$\alpha_1 > \overline{\alpha_1}$,
$\alpha_2 > \overline{\alpha_2}$,
$\overline{\alpha_3} > \alpha_3$,
$\overline{\alpha_4} > \alpha_4$,
$\overline{\alpha_1} + \alpha_4 > \alpha_4$,
$\overline{\alpha_2} + \alpha_3 > \alpha_3$.
It follows that the smallest angle strictly increases. □

What happens in the case where the Delaunay triangulation is not unique? The following still holds.

**Theorem 4.12** Let $P \subseteq \mathbb{R}^2$ be a finite set of points, not all on a line. Every Delaunay triangulation $D$ of $P$ maximizes the smallest angle among all triangulations $T$ of $P$.

**Proof.** Let $D$ be some Delaunay triangulation of $P$. We infinitesimally perturb the points in $P$ such that no four are on a common circle anymore. Then the Delaunay triangulation becomes unique (Theorem 4.10). Starting from $D$, we keep applying Lawson flips until we reach the unique Delaunay triangulation $D'$ of the perturbed point set. Now we examine this sequence of flips on the original *unperturbed* point set. All these flips must involve four cocircular points (only in the cocircular case, an infinitesimal perturbation can change “good” edges into “bad” edges that still need to be flipped). But as Figure 4.10 (a) easily implies, such a “degenerate” flip does not change the smallest of the six involved angles. It follows that $D$ and $D'$ have the same smallest angle, and since $D'$ maximizes the smallest angle among all triangulations $T$ (Theorem 4.11), so does $D$. □

**Questions**

10. **What is a triangulation of a point set?** Give a precise definition.

11. **Does every point set (not all points on a common line) have a triangulation?** You may for example argue with the scan triangulation.

12. **What is a Delaunay triangulation of a set of points?** Give a precise definition.

13. **What is the Delaunay graph of a point set?** Give a precise definition and a characterization.

14. **How can you prove that every set of points (not all on a common line) has a Delaunay triangulation?** You can for example sketch the Lawson flip algorithm and the Lifting Map, and use these to show the existence.

15. **When is the Delaunay triangulation of a point set unique?** Show that no 4 points being cocircular is a sufficient condition. Is this condition also necessary?

16. **What can you say about the “quality” of a Delaunay triangulation?** Prove that every Delaunay triangulation maximizes the smallest interior angle in the triangulation, among the set of all triangulations of the same point set.