Chapter 2

Convex Hull

2.1 Convexity

Consider $P \subset \mathbb{R}^d$. The following terminology should be familiar from linear algebra courses.

Linear hull.

$$\mathrm{lin}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \forall\, i : p_i \in P, \lambda_i \in \mathbb{R} \right\}$$

(smallest linear subspace containing P). For instance, if $P = \{p\} \subset \mathbb{R}^2 \setminus \{0\}$ then $\mathrm{lin}(P)$ is the line through p and the origin.

Affine hull.

$$\mathrm{aff}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \sum \lambda_i = 1 \wedge \forall\, i : p_i \in P, \lambda_i \in \mathbb{R} \right\}$$

(smallest affine subspace containing P). For instance, if $P = \{p, q\} \subset \mathbb{R}^2$ and $p \neq q$ then $\mathrm{aff}(P)$ is the line through p and q.

Convex hull.

Definition 2.1 *A set $P \subseteq \mathbb{R}^d$ is* **convex** *if and only if $\overline{pq} \subseteq P$, for any $p, q \in P$.*

Theorem 2.2 *A set $P \subseteq \mathbb{R}^d$ is convex if and only if $\sum_{i=1}^{n} \lambda_i p_i \in P$, for all $n \in \mathbb{N}$, $p_1, \dots, p_n \in P$, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.*

Proof. "$\Leftarrow$": obvious with $n = 2$.

"$\Rightarrow$": Induction on n. For $n = 1$ the statement is trivial. For $n \geq 2$, let $p_i \in P$ and $\lambda_i \geq 0$, for $1 \leq i \leq n$, and assume $\sum_{i=1}^{n} \lambda_i = 1$. We may suppose that $\lambda_i > 0$, for all i. (Simply omit those points whose coefficient is zero.)

Define $\lambda = \sum_{i=1}^{n-1} \lambda_i$ and for $1 \leq i \leq n-1$ set $\mu_i = \lambda_i/\lambda$. Observe that $\mu_i \geq 0$ and $\sum_{i=1}^{n-1} \mu_i = 1$. By the inductive hypothesis, $q := \sum_{i=1}^{n-1} \mu_i p_i \in P$, and thus by convexity of P also $\lambda q + (1-\lambda)p_n \in P$. We conclude by noting that $\lambda q + (1-\lambda)p_n = \lambda \sum_{i=1}^{n-1} \mu_i p_i + \lambda_n p_n = \sum_{i=1}^{n} \lambda_i p_i$. □

Observation 2.3 *For any family $(P_i)_{i\in I}$ of convex sets the intersection $\bigcap_{i\in I} P_i$ is convex.*

Definition 2.4 *The* **convex hull** $\mathrm{conv}(P)$ *of a set $P \subset \mathbb{R}^d$ is the intersection of all convex supersets of P.*

By Observation 2.3, the convex hull is convex, indeed.

Theorem 2.5 *For any $P \subseteq \mathbb{R}^d$ we have*

$$\mathrm{conv}(P) = \left\{ \sum_{i=1}^{n} \lambda_i p_i \;\middle|\; n \in \mathbb{N} \wedge \sum_{i=1}^{n} \lambda_i = 1 \wedge \forall i \in \{1,\dots,n\} : \lambda_i \geq 0 \wedge p_i \in P \right\} .$$

Proof. "$\supseteq$": Consider a convex set $C \supseteq P$. By Theorem 2.2 the right hand side is contained in C. As C was arbitrary, the claim follows.

"$\subseteq$": Denote the set on the right hand side by R. We show that R forms a convex set. Let $p = \sum_{i=1}^{n} \lambda_i p_i$ and $q = \sum_{i=1}^{n} \mu_i p_i$ be two convex combinations. (We may suppose that both p and q are expressed over the same p_i by possibly adding some terms with a coefficient of zero.)

Then for $\lambda \in [0,1]$ we have $\lambda p + (1-\lambda)q = \sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\mu_i)p_i \in R$, as $\sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\mu_i) = \lambda + (1-\lambda) = 1$. □

Definition 2.6 *The convex hull of a finite point set $P \subset \mathbb{R}^d$ forms a* **convex polytope**. *Each $p \in P$ for which $p \notin \mathrm{conv}(P \setminus \{p\})$ is called a* **vertex** *of $\mathrm{conv}(P)$. A vertex of $\mathrm{conv}(P)$ is also called an* **extremal point** *of P.*

Essentially, the following theorem shows that the term vertex above is well defined.

Theorem 2.7 *A convex polytope in $\mathbb{R}^d$ is the convex hull of its vertices.*

Proof. Let $P = \{p_1, \dots, p_n\}$, $n \in \mathbb{N}$, such that without loss of generality $p_1, \dots, p_k$ are the vertices of $\mathcal{P} := \mathrm{conv}(P)$. We prove by induction on n that $\mathrm{conv}(p_1, \dots, p_n) \subseteq \mathrm{conv}(p_1, \dots, p_k)$. For $n = k$ the statement is trivial.

For $n > k$, p_n is not a vertex of $\mathcal{P}$ and hence p_n can be expressed as a convex combination $p_n = \sum_{i=1}^{n-1} \lambda_i p_i$. Thus for any $x \in \mathcal{P}$ we can write $x = \sum_{i=1}^{n} \mu_i p_i = \sum_{i=1}^{n-1} \mu_i p_i + \mu_n \sum_{i=1}^{n-1} \lambda_i p_i = \sum_{i=1}^{n-1} (\mu_i + \mu_n \lambda_i) p_i$. As $\sum_{i=1}^{n-1} (\mu_i + \mu_n \lambda_i) = 1$, we conclude inductively that $x \in \mathrm{conv}(p_1, \dots, p_k)$. □

Theorem 2.8 (Carathéodory [3]) *For any $P \subset \mathbb{R}^d$ and $q \in \mathrm{conv}(P)$ there exist $k \leq d+1$ points $p_1, \dots, p_k \in P$ such that $q \in \mathrm{conv}(p_1, \dots, p_k)$.*

Theorem 2.9 (Separation Theorem) *Any two compact convex sets* $C, D \subset \mathbb{R}^d$ *with* $C \cap D = \emptyset$ *can be separated strictly by a hyperplane, that is, there exists a hyperplane* h *such that* C *and* D *lie in the opposite open halfspaces bounded by* h.

Proof. Consider the distance function $d : C \times D \to \mathbb{R}$ with $(c, d) \mapsto ||c - d||$. Since $C \times D$ is compact and d is continuous and strictly bounded from below by 0, d attains its minimum at some point $(c_0, d_0) \in C \times D$ with $d(c_0, d_0) > 0$. Let h be the hyperplane perpendicular to the line segment $\overline{c_0 d_0}$ and passing through the midpoint of c_0 and d_0.

If there was a point, say, c' in $C \cap h$, then by convexity of C the whole line segment $c_o c'$ lies in C and some point along this segment is closer to d_0 than is c_0, in contradiction to the choice of c_0. If, say, C has points on both sides of h, then by convexity of C it has also a point on h, but we just saw that there is no such point. Therefore, C and D must lie in different open halfspaces bounded by h. □

Actually, the statement above holds for arbitrary (not necessarily compact) convex sets, but the separation is not necessarily strict (the hyperplane may have to intersect the sets) and the proof is a bit more involved (cf. [7], but also check the errata on Jirka's webpage).

Altogether we obtain various equivalent definitions for the convex hull, summarized in the following theorem.

Theorem 2.10 *For a compact set* $P \subset \mathbb{R}^d$ *we can characterize* $\text{conv}(P)$ *equivalently as one of*

(a) the smallest (w. r. to set inclusion) convex subset of $\mathbb{R}^d$ *that contains* P*;*

(b) the set of all convex combinations of points from P*;*

(c) the set of all convex combinations formed by $d + 1$ *or fewer points from* P*;*

(d) the intersection of all convex supersets of P*;*

(e) the intersection of all closed halfspaces containing P.

Note that compactness is needed for (d) $\iff$ (e) only.

2.2 The convex hull problem in $\mathbb{R}^2$

Convex hull

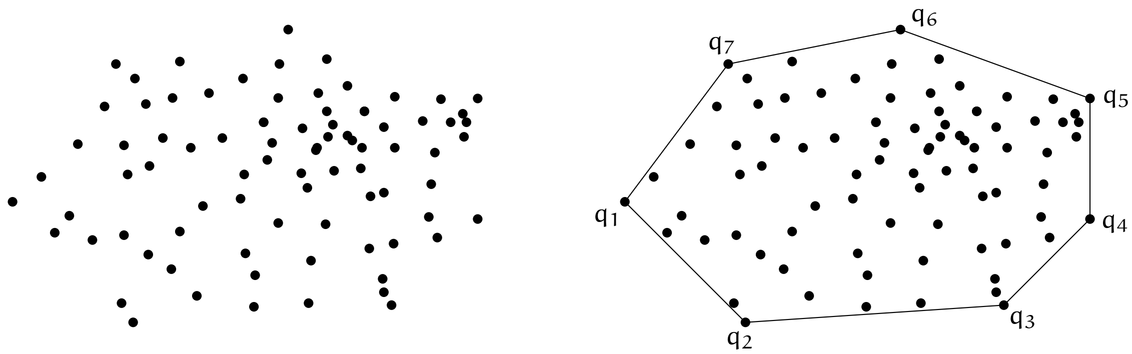
Input: $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Sequence $(q_1, \dots, q_h)$, $1 \le h \le n$, of the vertices of $\text{conv}(P)$ (ordered counterclockwise).

Extremal points

Input: $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Set $Q \subseteq P$ of the vertices of $\text{conv}(P)$.



(a) Input. (b) Output.

Figure 2.1: *Convex Hull of a set of points in* $\mathbb{R}^2$.

Degeneracies. A couple of further clarifications regarding the above problem definitions are in line.

First of all, for efficiency reasons an input is usually specified as a sequence of points. Do we insist that this sequence forms a set or are duplications of points allowed?

What if three points are collinear? Are all of them considered extremal? According to our definition from above, they are not and that is what we will stick to. But note that there may be cases where one wants to include all such points, nevertheless.

By the Separation Theorem, every extremal point p can be separated from the convex hull of the remaining points by a halfplane. If we take such a halfplane and shift its defining line such that it passes through p, then all points from P other than p should lie in the resulting open halfplane. In $\mathbb{R}^2$ we will work with the following definition.

Definition 2.11 *A point* $p \in P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ *is* **extremal** *for* $P \iff$ *there is a directed line* g *through* p *such that* $P \setminus \{p\}$ *is to the left of* g.

2.3 Trivial algorithms

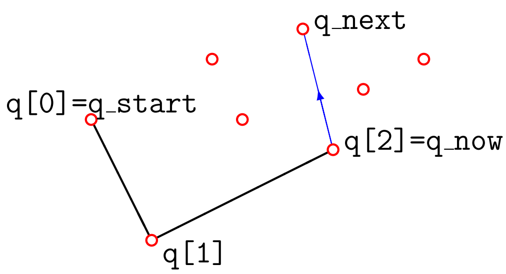
One can compute the extremal points using Carathéodory's Theorem as follows: Test for every point $p \in P$ whether there are $q, r, s \in P \setminus \{p\}$ such that p is inside the triangle with vertices q, r, and s. Runtime $O(n^4)$.

Another option, inspired by the Separation Theorem: test for every pair $(p, q) \in P^2$ whether all points from $P \setminus \{p, q\}$ are to the left of the directed line through p and q (or on the line segment $\overline{pq}$). Runtime $O(n^3)$.

2.4 Jarvis' Wrap

Find a point p_1 that is a vertex of conv(P) (e.g., the one with smallest x-coordinate). "Wrap" P starting from p_1, i.e., always find the next vertex of conv(P) as the one that is rightmost with respect to the previous vertex.

Besides comparing x-coordinates, the only geometric primitive needed is an *orientation* test: Denote by rightturn(p, q, r), for three points $p, q, r \in \mathbb{R}^2$, the predicate that is true if and only if r is (strictly) to the right of the oriented line pq.



Code for Jarvis' Wrap.

p[0..N) contains a sequence of points.
p_start point with smallest x-coordinate.
q_next some *other* point in p[0..N).

```
int h = 0;
Point_2 q_now = p_start;
do {
  q[h] = q_now;
  h = h + 1;

  for (int i = 0; i < N; i = i + 1)
    if (rightturn_2(q_now, q_next, p[i]))
      q_next = p[i];

  q_now = q_next;
  q_next = p_start;
} while (q_now != p_start);
```

q[0,h) describes a convex polygon bounding the convex hull of p[0..N).

Analysis. For every output point the above algorithm spends n rightturn tests, which is $\Rightarrow O(nh)$ in total.

Theorem 2.12 [6] *Jarvis' Wrap computes the convex hull of n points in $\mathbb{R}^2$ using $O(nh)$ rightturn tests, where h is the number of hull vertices.*

In the worst case we have $h = n$, that is, $O(n^2)$ rightturn tests. Jarvis' Wrap has a remarkable property that is called *output dependence*: the runtime depends not only on the size of the input but also on the size of the output. For a huge point set it constructs the convex hull in optimal linear time, if the convex hull consists of a constant number of vertices only. Unfortunately the worst case performance of Jarvis' Wrap is suboptimal, as we will see soon.

Degeneracies. The algorithm may have to cope with various degeneracies.

- Several points have smallest x-coordinate $\Rightarrow$ lexicographic order:

$$(p_x, p_y) < (q_x, q_y) \iff p_x < q_x \vee p_x = q_x \wedge p_y < q_y \; .$$

- Three or more points collinear $\Rightarrow$ choose the point that is farthest among those that are rightmost.

Predicates. Besides the lexicographic comparison mentioned above, the Jarvis' Wrap (and most other 2D convex hull algorithms for that matter) need one more geometric predicate: the rightturn or—more generally—orientation test. The computation amounts to evaluating a polynomial of degree two, as the proposition given below shows. We therefore say that the orientation test has *algebraic degree* two. In contrast, the lexicographic comparison has degree one only. The algebraic degree not only has a direct impact on the efficiency of a geometric algorithm (lower degree $\leftrightarrow$ less multiplications), but also an indirect one because high degree predicates may create large intermediate results, which may lead to overflows and are much more costly to compute with exactly.

Proposition 2.13 *For three points* $(p_x, p_y), (q_x, q_y), (r_x, r_y) \in \mathbb{R}^2$*, the sign of the determinant*

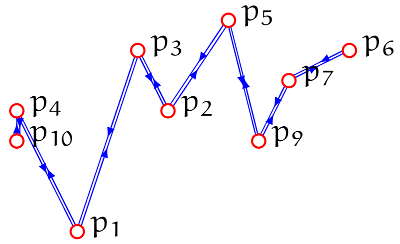
$$\begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

determines if r *lies to the right, to the left or on the directed line through* p *and* q*.*

Proof. Exercise. □

2.5 Graham Scan (Successive Local Repair)

Sort points lexicographically and remove duplicates: $(p_1, \ldots, p_n)$.



$$p_{10}\, p_4\, p_1\, p_3\, p_2\, p_5\, p_9\, p_7\, p_6\, p_7\, p_9\, p_5\, p_2\, p_3\, p_1\, p_4\, p_{10}$$

As long as there is a (consecutive) triple (p, q, r) s.t. q is left of or on the directed line $\overrightarrow{pr}$, remove q from the sequence.

Code for Graham Scan.

`p[0..N)` lexicographically sorted sequence of pairwise distinct points, $N \geq 2$.

```
q[0] = p[0];
int h = 0;
// Lower convex hull (left to right):
for (int i = 1; i < N; i = i + 1) {
  while (h>0 && rightturn_2(q[h-1], q[h], p[i]))
    h = h - 1;
  h = h + 1;
  q[h] = p[i];
}

// Upper convex hull (right to left):
for (int i = N-2; i >= 0; i = i - 1) {
  while (rightturn_2(q[h-1], q[h], p[i]))
    h = h - 1;
  h = h + 1;
  q[h] = p[i];
}
```

`q[0,h)` describes a convex polygon bounding the convex hull of `p[0..N)`.

Analysis.

Theorem 2.14 *The convex hull of a set $P \subset \mathbb{R}^2$ of n points can be computed using $O(n \log n)$ geometric operations.*

Proof.

1. Sorting and removal of duplicate points: $O(n \log n)$.

2. At begin: $2n - 2$ points; at the end: h points. $\Rightarrow 2n - h - 2$ shortcuts/positive orientation tests. In addition at most $2n - 2$ negative tests. Altogether at most $4n - h - 4$ orientation tests.

In total $O(n \log n)$ operations. Note that the number of orientation tests is linear only, but $O(n \log n)$ lexicographic comparisons are needed. □

There are many variations of this algorithm, the basic idea is due to Graham [5].

2.6 Lower Bound

Theorem 2.15 $\Omega(n \log n)$ *geometric operations are needed to construct the convex hull of* n *points in* $\mathbb{R}^2$ *(in the algebraic computation tree model).*

Proof. Reduction from sorting (for which it is known that $\Omega(n \log n)$ comparisons are needed in the algebraic computation tree model). Given n real numbers $x_1, \ldots, x_n$, construct a set $P = \{p_i \mid 1 \le i \le n\}$ of n points in $\mathbb{R}^2$ by setting $p_i = (x_i, x_i^2)$. This construction can be regarded as embedding the numbers into $\mathbb{R}^2$ along the x-axis and then projecting the resulting points vertically onto the unit parabola. The order in which the points appear along the lower convex hull of P corresponds to the sorted order of the x_i. Therefore, if we could construct the convex hull in $o(n \log n)$ time, we could also sort in $o(n \log n)$ time. □

Clearly this simple reduction does not work for the Extremal Points problem. But using a more involved construction one can show that $\Omega(n \log n)$ is also a lower bound for the number of operations needed to compute the set of extremal points only. This was first shown by Avis [1] for linear computation trees, then by Yao [8] for quadratic computation trees, and finally by Ben-Or [2] for general algebraic computation trees.

In fact, the argument is based on a lower bound of $\Omega(n \log n)$ operations for *Element Uniqueness*: Given n real numbers, are any two of them equal? At first glance, this problem appears a lot easier than sorting, but apparently it is not, at least in this model of computation.

2.7 Chan's Algorithm

Given matching upper and lower bounds we may be tempted to consider the algorithmic complexity of the planar convex hull problem settled. However, this is not really the case: Recall that the lower bound is a worst case bound. For instance, the Jarvis' Wrap runs in $O(nh)$ time an thus beats the $\Omega(n \log n)$ bound in case that $h = o(\log n)$. The question remains whether one can achieve both output dependence and optimal worst case performance at the same time. Indeed, Chan [4] presented an algorithm to achieve this runtime by cleverly combining the "best of" Jarvis' Wrap and Graham Scan. Let us look at this algorithm in detail.

Divide. *Input:* a set $P \subset \mathbb{R}^2$ of n points and a number $H \in \{1, \ldots, n\}$.

1. Divide P into $k = \lceil n/H \rceil$ sets $P_1, \ldots, P_k$ with $|P_i| \le H$.

2. Construct $\mathrm{conv}(P_i)$ for all i, $1 \le i \le k$.

3. Construct H vertices of $\mathrm{conv}(P)$. (*conquer*)

Analysis. Step 1 takes $O(n)$ time. Step 2 can be handled using Graham Scan in $O(H \log H)$ time for any single P_i, that is, $O(n \log H)$ time in total.

Conquer.

1. Find the lexicographically smallest point in $\text{conv}(P_i)$ for all i, $1 \leq i \leq k$.
2. Starting from the lexicographically smallest point of P find the first H points of $\text{conv}(P)$ oriented counterclockwise (simultaneous Jarvis' Wrap on the sequences $\text{conv}(P_i)$).

Determine in every step the points of tangency from the current point of $\text{conv}(P)$ to $\text{conv}(P_i)$, $1 \leq i \leq k$, using binary search.

Analysis. Step 1 takes $O(n)$ time. Step 2 consists of at most H wrap steps. Each wrap needs to find the minimum among k candidates where each candidate is computed by a binary searches on at most H elements. This amounts to $O(Hk \log H) = O(n \log H)$ time for Step 2.

Remark. Using a more clever search strategy instead of many binary searches one can handle the conquer phase in $O(n)$ time. However, this is irrelevant as far as the asymptotic runtime is concerned, given that already the divide step takes $O(n \log H)$ time.

Searching for h. While the runtime bound for $H = h$ is exactly what we were heading for, it looks like in order to actually run the algorithm we would have to know h, which—in general—we do not. Fortunately we can circumvent this problem rather easily, by applying what is called a *doubly exponential search.* It works as follows.

Call the algorithm from above iteratively with parameter $H = \min\{2^{2^t}, n\}$, for $t = 0, \ldots$, until the conquer step finds all extremal points of P (i.e., the wrap returns to its starting point).

Analysis: Let 2^{2^s} be the last parameter for which the algorithm is called. Since the previous call with $H = 2^{2^{s-1}}$ did not find all extremal points, we know that $2^{2^{s-1}} < h$, that is, $2^{s-1} < \log h$, where h is the number of extremal points of P. The total runtime is therefore at most

$$\sum_{i=0}^{s} cn \log 2^{2^i} = \sum_{i=0}^{s} cn2^i = cn(2^{s+1} - 1) < 4cn \log h = O(n \log h).$$

In summary, we obtain the following theorem.

Theorem 2.16 *The convex hull of a set* $P \subset \mathbb{R}^2$ *of* n *points can be computed using* $O(n \log h)$ *geometric operations, where* h *is the number of convex hull vertices.*

Questions

1. *How is convexity defined? What is the convex hull of a set in* $\mathbb{R}^d$*?* Give at least three possible definitions.

2. *What does it mean to compute the convex hull of a set of points in* $\mathbb{R}^2$*?* Discuss input and expected output and possible degeneracies.

3. *How can the convex hull of a set of* n *points in* $\mathbb{R}^2$ *be computed efficiently?* Describe and analyze (incl. proofs) Jarvis' Wrap, Successive Local Repair, and Chan's Algorithm.

4. *Is there a linear time algorithm to compute the convex hull of* n *points in* $\mathbb{R}^2$*?* Prove the lower bound and define/explain the model in which it holds.

5. *Which geometric primitive operations are needed to compute the convex hull of* n *points in* $\mathbb{R}^2$*?* Explain the two predicates and how to compute them.

References

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