When is a graph planar?

**Theorem** (Euler, 1758) If a plane multigraph $G$ with $k$ components has $n$ vertices, $e$ edges, and $f$ faces, then

$$n - e + f = 1 + k.$$  

**Corollary** If $G$ is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.

If $G$ is also triangle-free, then $e(G) \leq 2n(G) - 4$.

**Corollary** $K_5$ and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacement of $e$ with a new vertex $z$ and two new edges $xz$ and $zy$. The graph $H'$ is a subdivision of $H$, if one can obtain $H'$ from $H$ by a series of edge subdivisions. Vertices of $H'$ with degree at least three are called branch vertices.

**Theorem** (Kuratowski, 1930) A graph $G$ is planar iff $G$ does not contain a subdivision of $K_5$ or $K_{3,3}$. 
Kuratowski’s Theorem

**Theorem** (Kuratowski, 1930) A graph $G$ is planar iff $G$ does not contain a subdivision of $K_5$ or $K_{3,3}$.

*Proof.*
A Kuratowski subgraph of $G$ is a subgraph of $G$ that is a subdivision of $K_5$ or $K_{3,3}$. A minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

A counterexample to Kuratowski’s Theorem constitutes a nonplanar graph that does not contain any Kuratowski subgraph.

Kuratowski’s Theorem follows from the following Main Lemma and Theorem.
Main Lemma. If $G$ is a graph with fewest edges among counterexamples, then $G$ is 3-connected.

**Lemma 1.** Every minimal nonplanar graph is 2-connected.

**Lemma 2.** Let $S = \{x, y\}$ be a separating set of $G$. If $G$ is a nonplanar graph, then adding the edge $xy$ to some $S$-lobe of $G$ yields a nonplanar graph.

Main Theorem. (Tutte, 1960) If $G$ is a 3-connected graph with no Kuratowski subgraph, then $G$ has a convex embedding in the plane with no three vertices on a line.

A convex embedding of a graph is a planar embedding in which each face boundary is a convex polygon.

**Lemma 3.** If $G$ is a 3-connected graph with $n(G) \geq 5$, then there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

*Notation:* $G \cdot e$ denotes the graph obtained from $G$ after the contraction of edge $e$.

**Lemma 4.** $G$ has no Kuratowski subgraph $\Rightarrow G \cdot e$ has no Kuratowski subgraph.
Proof of Tutte’s Theorem

**Main Theorem.** (Tutte, 1960) If $G$ is a 3-connected graph with no Kuratowski subgraph, then $G$ has a convex embedding in the plane with no three vertices on a line.

**Proof.** Induction on $n(G)$. 
Base case: $G$ is 3-connected, $n(G) = 4$ \(\Rightarrow\) $K_4$.

Let $e \in E$ be s.t. $H = G \cdot e$ is 3-connected. (Lemma 3) Then $H$ has no Kuratowski subgraph. (Lemma 4) Induction \(\Rightarrow\) $H$ has a convex embedding in the plane with no three vertices on a line.

Let $z \in V(H)$ be the contracted $e$.
$H - z$ is 2-connected \(\Rightarrow\) boundary of the face containing $z$ after the deletion of the edges incident to $z$ is a cycle $C$. 

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Let $x_0, \ldots, x_{k-1}$ be the neighbors of $x$ on $C$ in cyclic order. Note that $|N(x)| \geq 3$ and hence $k \geq 1$.

Denote by $\langle x_i, x_{i+1} \rangle$ the portion of $C$ from $x_i$ to $x_{i+1}$ (including endpoints; indices taken modulo $k$.)

Let $N_x = N(x) \setminus \{y\}$ and $N_y = N(y) \setminus \{x\}$.

**Case 1.** $|N_x \cap N_y| \geq 3$.

Let $u, v, w \in N_x \cap N_y$. Then $x, y, u, v, w$ are the branch vertices of a $K_5$-subdivision in $G$.

**Case 2.** $|N_x \cap N_y| \leq 2$.

Since $|N_x \cup N_y| \geq 3$, there is w.l.o.g. a vertex $u \in N_y \setminus N_x$. Let $i$ be such that $u$ is on $\langle x_i, x_{i+1} \rangle$.

**Case 2a.** $N_y$ is contained in $\langle x_i, x_{i+1} \rangle$.

Then there is an appropriate embedding of $G$: Placing $x$ in place of $z$ and $y$ sufficiently close to $x$ maintains convexity. (No three vertices are collinear).

**Case 2b.** For every $i$ there is a vertex in $N_y$ that is not contained in $\langle x_i, x_{i+1} \rangle$.

Then there must be a $v \in N_y$ that is not on $\langle x_i, x_{i+1} \rangle$ and $x, y, x_i, x_{i+1}, u, v$ are the branch vertices of a $K_{3,3}$-subdivision in $G$. 
Proof of the Lemmas

Lemma 3. $G$ is 3-connected, $n(G) \geq 5 \Rightarrow$ there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Proof. Suppose $G$ is 3-connected and for every $e \in E$, $G \cdot e$ is NOT 3-connected.

For edge $e = xy$, the vertex $z$ is a mate of $xy$ if $\{x, y, z\}$ is a cut in $G$.

Choose $e = xy$ and their mate $z$ such that $G - \{x, y, z\}$ has a component $H$, whose order is as large as possible.

Let $H'$ be another component of $G - \{x, y, z\}$ and let $u \in V(H')$ be a neighbor of $z$ (there is one). Let $v$ be a mate of $uz$.

$V(H) \cup \{x, y\} \setminus \{v\}$ is connected in $G - \{z, u, v\}$ contradicting the maximality of $H$. \qed
Lemma 4. $G$ has no Kuratowski subgraph $\Rightarrow G \cdot e$ has no Kuratowski subgraph.

Proof. Suppose $G \cdot e$ contains a Kuratowski subgraph $H$. Then

- $z \in V(H)$
- $z$ is a branchvertex of $H$
- $|N_H(z)| = 4$ and $|N_H(z) \cap N_G(x)|, |N_H(z) \cap N_G(y)| \geq 2$

Then $H$ is the subdivision of $K_5 \Rightarrow G$ contains a subdivision of $K_{3,3}$, a contradiction. $\square$
Minors

$K_7$ is a toroidal graph (it is embeddable on the torus), $K_8$ is not. What else is not? For the torus there is NO equivalent version of Kuratowski’s characterization with a finite number of forbidden subdivisions. Any such characterization must lead to an infinite list.

A weaker concept: Minors.

Graph $G$ is called a minor of graph $H$ if $G$ can be obtained from $H$ by a series of edge deletions and edge contractions. Graph $H$ is also called a $G$-minor

Example: $K_5$ is a minor of the Petersen graph $P$, but $P$ does not contain a $K_5$-subdivision.
The Graph Minor Theorem

**Theorem.** (Robertson and Seymour, 1985-2005) In any infinite list of graphs, some graph is a minor of another.

**Proof:** more than 500 pages in 20 papers.

**Corollary** For any graph property that is closed under taking minors, there exists finitely many minimal forbidden minors.

**Remark:** Wagner’s Theorem, stating that every non-planar graph contains either a $K_5$ or $K_{3,3}$-minor, can be (quite straightforwardly) deduced from Kuratowski’s Theorem.

For embeddability on the projective plane, it is known that there are 35 minimal forbidden minors. For embeddability on the torus, we don’t know the exact number of minimal forbidden minors; there are more than 800 known.