Graphs & Algorithms: Advanced Topics

Treewidth

Uli Wagner

ETH Zürich

Dynamic Programming on Trees

Example

MaxIndependentSet in a tree $T = (V, E)$ rooted at $r \in V$.

For $v \in V$ let T_v denote the subtree rooted at v.

$$f^+(v) := \max\{|I| : I \subseteq V(T_v) \text{ independent}, v \in I\}$$

$$f^-(v) := \max\{|I| : I \subseteq V(T_v) \text{ independent}, v \notin I\}$$

Algorithm Compute maximum independent set in T (time $O(|V|)$)

Traverse T in post order starting from r. $v :=$ current vertex

- If v is a leaf, let $f^+(v) = 1$ and $f^-(v) = 0$.
- Else let $x_1, \ldots, x_k$ be the children of v

$$f^+(v) := 1 + \sum_{i=1}^{k} f^-(x_i), \quad f^-(v) := \sum_{i=1}^{k} \max\{f^+(x_i), f^-(x_i)\}$$

Return $\max\{f^+(r), f^-(r)\}$.

Tree Decompositions

Definition

A tree decomposition for a graph $G = (V, E)$ is a pair

$$(\,\underbrace{\{X_i \mid i \in I\}}_{\text{bags}},\quad \underbrace{T = (I, F)}_{\text{tree}}\,)$$

such that

- $\bigcup_{i \in I} X_i = V$ (bags cover vertices);
- $\{u, v\} \in E \Rightarrow \{u, v\} \subseteq X_i$ for some $i \in I$ (bags cover edges);
- for all $v \in V$ the set $I_v = \{i \in I \mid v \in X_i\}$ is connected in T (tree property)

The width of a tree decomposition is $\max_{i \in I} |X_i| - 1$.

The treewidth of G is the minimum width of a tree decomposition for G, denoted $\text{tw}(G)$

Example. Trees have treewidth 1.

Basic Observations

Observation

For any graph $G = (V, E)$ a single bag containing V forms a tree decomposition of width $n - 1$.

Goal

Tree decompositions of small width, certify that the graph is in some way "tree-like"

Observation

If $H \subseteq G$ is a subgraph then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$

Proposition

If a graph $G = (V, E)$ has two components A and B with $A \cup B = V$ then $\mathrm{tw}(G) = \max\{\mathrm{tw}(A), \mathrm{tw}(B)\}$.

Treewidth of cliques and grids

Lemma

Let $(\{X_i \mid i \in I\}, T = (I, F))$ be a tree decomposition for G. For any clique $G[W]$, $W \subseteq V(G)$, there is an $i \in I$ such that $W \subseteq X_i$.

Proof. Root T arbitrarily. For $w \in W$ choose a "bag label" $r_w \in I_w$ of minimal depth (distance from the root). Then the bag with the label $\{r_w \mid w \in W\}$ with maximum depth contains W. □

Corollary

The treewidth of K_n is $n - 1$.

Example

The $n \times n$-grid on $\{(i, j) \mid 1 \leq i, j \leq n\}$ has treewidth $\leq n$:
Consider the path on

$$X_{n(i-1)+j} = \{(i, k) \mid j \leq k \leq n\} \cup \{(i + 1, k) \mid 1 \leq k \leq j\},$$

$1 \leq i \leq n - 1,\ 1 \leq j \leq n.$

How many vertices are needed in T?

Definition

A tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of width k is smooth if

- $|X_i| = k + 1$ for all $i \in I$;
- $|X_i \cap X_j| = k$ for all $\{i, j\} \in F$.

Proposition

For any graph with treewidth k there exists a smooth tree decomposition of width k. $\rightarrow$ *Exercise*

Lemma

If $(X, T = (I, F))$ is a smooth tree decomposition of width k for $G = (V, E)$ then $|I| = |V| - k$. In particular, $n(T) \leq n(G)$.

Proof. Induction on $|I|$. Consider a leaf $i \Rightarrow$ unique vertex $v \in X_i$ that does not belong to any other X_j, $j \neq i$. Removing i from T yields smooth tree decomposition of $G[V \setminus \{v\}]$. □

Number of edges

Lemma

A graph $G = (V, E)$ of treewidth at most k has at most $k|V| - \binom{k+1}{2}$ edges.

Proof. Induction on $|V|$. Base case is $|V| = k + 1$. Consider a smooth tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ for G and a leaf i of T. Then there is a unique vertex $v \in X_i$ that does not belong to any other X_j, $j \neq i$. Clearly $\deg_G(v) \leq k$. Removing i from T yields a tree decomposition for $G[V \setminus \{v\}]$. □

Corollary

A graph has treewidth 1 if and only if it is a forest (and contains at least one edge).

Treewidth and Cuts

Lemma

Let $(\{X_i \mid i \in I\}, T = (I, F))$ be a tree decomposition for a connected graph $G = (V, E)$ such that $X_i \not\subseteq X_j$ for $i \neq j$. Then

a) *$X_i \cap X_j$ is a vertex cut in G for any $\{i, j\} \in F$, i.e., $G - (X_i \cap X_j)$ is disconnected.*

b) *X_i is a vertex cut in G for any $i \in I$ that is not a leaf in T.*

Remark

It is possible to adapt any tree decomposition in $O(|I|)$ time to fulfill the non-containment condition without changing its width. (If there is a containment, there is also one between adjacent bags.)

Treewidth and Separators

Theorem

Tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ *of width* k *for* $G \Rightarrow$ *can compute* $(k+1, \frac{1}{2})$*-separator for* G *in* $O(|I|)$ *time.*

Proof. Root T arbitrarily and define a weight function w on I by $w(i) := |X_i \setminus X_{\mathrm{parent}(i)}|$. Each $v \in V$ is counted exactly once (bags containing v are connected) $\Rightarrow \sum_{i \in I} w(i) = |V|$.
By the Separator Theorem for (weighted) trees we obtain a $(1, \frac{1}{2})$-separator s for T.
Removing X_s disconnects G because

- any $v \in V \setminus X_s$ can appear in at most one subtree (otw, it would also appear in X_s by connectivity);
- each subtree defines at least one component (no edge between subtrees);
- each subtree (and hence component) consists of at most $\frac{n}{2}$ vertices. □

Dynamic Programming on Graphs of Treewidth at most k

Given: $G = (V, E)$ and smooth tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of width $\leq k$ for G

Algorithm to compute a *maximum independent set* in G in time $O(k^2 4^k |V|)$

Pick an arbitrary root $r \in I$ and for $i \in I$ let $V_i = \bigcup_{j \in T_i} X_j$, where T_i denotes the subtree rooted at i.

For $U \subseteq X_i$ let $f^U(i)$ be the size of a maximum independent subset of V_i whose intersection with X_i is exactly U.

- Traverse T starting from r top-down, Let $i :=$ current vertex.
- If i is a leaf, for every $U \subseteq X_i$, we have $f^U(i) = |U|$ if U is independent in G and $f^U(i) = -\infty$, otherwise.
- Else let $c_1, \ldots, c_\ell$ be the children of i. We have

$$f^U(i) = |U| + \sum_{j=1}^{\ell} \max \Big\{ f^W(c_j) - |U \cap W| \Big|$$

$$W \subseteq X_{c_j} \text{ and } W \cap X_i = U \cap X_{c_j} \text{ and } W \cup U \text{ independent} \Big\}$$

(Convention: $\max(\emptyset) = -\infty$.)

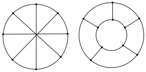
Treewidth and Minors

Proposition

Graphs of treewidth at most k are closed under taking minors.

Proof. Removal of edges and isolated vertices are trivial. When contracting an edge $\{u, v\}$, replace all occurrences of u in any bag by v. □

By the Robertson-Seymour Graph Minor Theorem there is hence a finite set of forbidden minors. But they are not known, except for small k.

- $k = 0$: K_2.
- $k = 1$: K_3.
- $k = 2$: K_4.
- $k = 3$: K_5, $K_{2,2,2}$, 
- $k = 4$: more than 75...

Computing treewidth

Theorem (Arnborg, Corneil, Proskurowski '87)

For given G and variable k, it is NP-complete to decide whether $\mathrm{tw}(G) \leq k$.

Theorem (Bodlaender '96)

For any fixed $k \in \mathbb{N}$ there exists a linear time algorithm to test whether a given graph has treewidth at most k and—if so—output a corresponding tree decomposition.
(The running time is exponential in k^3.)

Open Problem

- Is there a (polynomial-time) constant-factor approximation algorithm for treewidth? Is there a PTAS?
  (Known: $O(\sqrt{\log n})$-approximation algorithm for vertex separators [Feige, Hajiaghayi and Lee 2008] , implies the same for treewidth.)
- Can the treewidth be computed in polynomial time for planar graphs?

Not everything is easy for bounded treewidth...

Theorem (Nishizeki,Vygen,Zhou '01)

Edge-disjoint paths is NP-complete for graphs of treewidth 2.

(Trivial for trees and polynomial for outerplanar graphs.)

Input: a graph $G = (V, E)$ and pairs $\{s_i, t_i\} \in \binom{V}{2}$, $1 \leq i \leq k$
Goal: find k edge-disjoint paths P_i in G such that P_i connects s_i and t_i.

Theorem (McDiarmid & Reed '01)

Weighted coloring is NP-hard for graphs of treewidth 3.

(Trivial for bipartite graphs, hence for forests.)

Input: a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{N}$
Goal: a weighted k-coloring, i.e., a function $c : V \to [k]$ such that $|c(u) - c(v)| \geq w(e)$ for all $\{u, v\} \in E$.

Cops and robber

In the omniscient cops and robber game, k cops each occupy a vertex of a graph in which a robber moves around, trying to escape capture. The robber moves along edges "at infinite speed", the cops move "by helicopter".

Definition

Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, a position in the k cops and robber game on G is a pair (C, r), where

- $C \in \binom{V}{k}$ (location of cops) and
- r is a vertex in some component of $G \setminus C$ (location of robber).
- In Round 0, the cops choose $C_0 \in \binom{V}{k}$ and then the robber chooses $r_0 \in V \setminus C_0$ arbitrarily.
- In Round i, $i > 0$, the cops choose $C_i \in \binom{V}{k}$ and then the robber chooses a vertex $r_i \in V \setminus C_i$ such that there is a path between r_i and r_{i-1} in $G \setminus (C_i \cap C_{i-1})$.
- The cops win if after some finite number of rounds the robber has no vertex to choose.

Cops, Robber, and Treewidth

Theorem (Seymour & Thomas '93)

If a graph G has treewidth at most k then $k+1$ omniscient cops can catch a robber on G.

Proof. Suppose $n(G) > k+1$ and let $(\{X_i \mid i \in I\}, T = (I, F))$ be a smooth tree decomposition of width $\leq k$ for G.

- Pick an arbitrary root $a \in I$ and for $i \in I$ let $V_i = \bigcup_{j \in T_i} X_j$, where T_i denotes the subtree rooted at i.
- In the first round choose $C_0 = X_a$.
- In Round j, we suppose $C_{j-1} = X_b$ for some $b \in I$ and $r_{j-1} \in V_b \setminus X_b$. Let c be the child of b for which $r_{j-1} \in V_c$. Observe that $X_b \cap X_c$ is a k-cut in G. Thus choosing $C_j = X_c$ confines the robber to $V_c \setminus X_c$.
- After a finite number of steps, we arrive at a leaf ℓ of T for which $V_\ell \setminus X_\ell = \emptyset$. Thus, the robber has nowhere to go. □

Remark

The converse also holds but the proof is much more involved.

Cops and robber on the grid

Proposition

On the $n \times n$-grid $n-1$ omniscient cops cannot catch a robber.

Proof. Whichever positions the $n-1$ cops occupy, there are always a cop-free row and a cop-free column.

Claim. The robber can always move to the intersection of a cop-free row with a cop-free column.

Initially, this is clear. Suppose that at some point one or more cops enter the free row and/or free column where the robber is located. Then the robber can move along the previously free row to the to-be free column and within this column to the to-be free row. □

Proposition

On the $n \times n$-grid n omniscient cops cannot catch a robber, for $n \geq 2$. → *Exercise.*

Corollary

The $n \times n$-grid has treewidth n. $\Rightarrow$ There are planar graphs on n vertices whose treewidth is $\Omega(\sqrt{n})$.

Partial k-trees

Definition

A k-tree is a graph formed from a k-clique by iteratively joining a new vertex to some k-clique.

In other words, a graph is a k-tree $\iff$ there is an order $\pi = (v_1, v_2, \ldots, v_n)$ of its vertices such that the neighbors of v_i preceding it in π form a $\min\{i-1, k\}$-clique, for all $1 \leq i \leq n$.

Observation

a) *1-trees are exactly trees.*

b) *A k-tree on $n \geq k$ vertices has $kn - \binom{k+1}{2}$ edges.*

Definition

A graph is a partial k-tree if it is a subgraph of a k-tree.

Partial k-trees and treewidth

Theorem

A graph G is partial k-tree $\iff$ G has treewidth at most k.

Proof ("$\Leftarrow$"): Let $(\{X_i \mid i \in I\}, T = (I, F))$ be a smooth tree decomposition of width $\leq k$ for G. Add all edges inside bags to G.

Claim. The resulting graph H is a k-tree.

Induction on $|I|$:

- $|I| = 1$: H is K_{k+1}, a k-tree.
- Otherwise, let $i \in I$ be a leaf of T. Then there is a $v \in X_i$ that does not occur in any X_j, $j \in I \setminus \{i\}$. Removal of i from I results in a tree decomposition of width $\leq k$ for $H' = H - v$.
- By induction H' is a k-tree. Adding v to H' and connect it to the k-clique $X_i \setminus \{v\}$, we get H, which hence is a k-tree.

□

Partial k-trees and treewidth

Theorem

A graph G is partial k-tree $\iff$ G has treewidth at most k.

Proof ("$\Rightarrow$"): Let H be a k-tree containing G and $\pi = (v_1, \ldots, v_n)$ a vertex order for H such that the neighbors of v_i preceding it in π form a $\min\{i-1, k\}$-clique, for all $1 \le i \le n$.

Build a tree decomposition of width k for $V_i = \{v_1, \ldots, v_i\}$ inductively such that for every j, $1 \le j \le i$, there is a bag that contains $\{v_j\} \cup \Pi_j$, where $\Pi_j = V_j \cap N_H(v_j)$.

- $i \le k+1$: A single bag for V_i suffices.
- Otw, let $\ell = \max\{1 \le \ell < i \mid v_\ell \in \Pi_j\}$. By the induction hypothesis there is a tree decomposition of width k for V_{i-1} in which one bag X_a contains $\{v_\ell\} \cup \Pi_\ell$.
- Create a new node b, make it adjacent to a only, and set $X_b = \{v_i\} \cup \Pi_i$. (Note that $\Pi_i \subseteq \{v_\ell\} \cup \Pi_\ell$ because Π_i is a clique.) □

Grids, minors, and treewidth

Theorem (Alon, Seymour, Thomas '90)

For any fixed graph H, every graph G that does not contain H as a minor has treewidth at most $n(H)^{3/2}\sqrt{n(G)}$.

Corollary

A planar graph on n vertices has treewidth $O(\sqrt{n})$.

Theorem ([Robertson, Seymour, Thomas '94)

Every graph of treewidth larger than 20^{2k^5} has a $k \times k$-grid as a minor.
On the other hand, there are graphs of treewidth $\Omega(k^2 \log k)$ that do not have a $k \times k$-grid as a minor.