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Extremal Combinatorics — *SS 07*

Exercise Set 2

Exercise 1

Define $ex(n,m,H)$ as the largest number e, such that there is an H-free bipartite graph with partite sets of size n and m respectively containing e edges. Show that

$$ex(q^2+q+1, q^2+q+1, K_{2,2}) = (q^2+q+1)(q+1)$$

for every prime power q, i.e. the graph from Construction 1 is *an* optimal construction.

Exercise 2

The goal of this exercise is to calculate

$$N(\alpha) := |\{(x,y) \in \mathbb{F}_q^2 \,:\, x^2+y^2=\alpha\}|$$

in the field $\mathbb{F}_q$ (q being an odd prime tower) by going through the following steps:

(i) Determine $N(0)$ by distinguishing the cases $-1 \in QR(q)$ and $-1 \in QNR(q)$.

(ii) Show that $N(\alpha)$ is constant on $QR(q)$ and it is constant on $QNR(q)$.

(iii) Calculate $N(1)$ by double-counting the quantity $\sum_{w\in QR(q)} N(w)$.

(iv) For fixed $b \in QNR(q)$ calculate $N(b)$.

(v) Conclude that the number of edges of the graph from Construction 2 is in fact $n^{3/2}/2 - O(\sqrt{n})$.

Exercise 3

Define the *polarity graph* G (Construction 3) on the points of the projective plane $PG(p,2)$ in the following way. Let $V(G) = \{[x_0,x_1,x_2] \in \mathcal{P}\}$ and $E(G) = \{\{x,y\} : x \neq y, x_0y_0 + x_1y_1 + x_2y_2 = 0\}$.

Show that the polarity graph indeed contains exactly $p+1$ degree-p-vertices.