Eidgenössische Technische Hochschule Zürich | Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Dr. Tibor Szabó and Philipp Zumstein

20.04.07

Extremal Combinatorics — *SS 07*
Exercise Set 3

Exercise 1

Let $\mathbb{F}$ be an arbitrary finite field. Prove that if $-1 \in \mathbb{F}$ is a square, then the corresponding sphere-graph on n vertices (defined in the 3-dimensional space over $\mathbb{F}$) not only contains a $K_{3,3}$, but also a $K_{n^{1/3},n^{1/3}}$.

Exercise 2

Prove that the chromatic number of the unit-distance graph for the plane is between 4 and 7.

Exercise 3

Let q be any odd prime power. Recall that the equation $x^2 + y^2 = \beta$, where $\beta \neq 0$ is fixed, has $q - 1$ solutions $(x, y) \in \mathbb{F}_q^2$ if -1 is a quadratic residue in $\mathbb{F}_q$, and $q+1$ solutions if -1 is not a quadratic residue; furthermore, $x^2 + y^2 = 0$ has $2q - 1$ solutions if $-1 \in QR(q)$, or 1 single solution if $-1 \in QNR(q)$.

(a) Give a general exact formula for $N_k(\beta)$ the number of solutions to $x_1^2 + \cdots + x_k^2 = \beta$ for any fixed $k \in \mathbb{N}, \beta \in \mathbb{F}_q$.

(b) Give an elementary proof that for any $a \in \mathbb{F}_q^3$ the sphere $S_\alpha(a)$ contains either $q^2 - q$ or $q^2 + q$ points depending on whether α and -1 are quadratic residues or not.

(c) Count the number of edges in the Brown graph.