Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Dr. Tibor Szabó and Philipp Zumstein

11.05.07

Extremal Combinatorics — *SS 07*

Exercise Set 6

Exercise 1

Derive the Moore bound for even girth, i.e. show that if G is a d-regular graph with girth $2k$, then

$$n(G) \geq 2\sum_{i=0}^{k-1}(d-1)^i.$$

Exercise 2

The set of points $\mathcal{P}$ in $PG(q,4)$ — the projective 4-space over $\mathbb{F}_q$ — are the equivalence classes of $\mathbb{F}_q^5 \setminus \{(0,\ldots,0)\}$, where two 5-tuples are in relation if they are nonzero constant multiples of each other:

$$[x_{-2},x_{-1},x_0,x_1,x_2] := \Big\{(cx_{-2},cx_{-1},cx_0,cx_1,cx_2) \in \mathbb{F}_q^5 \setminus \{(0,\ldots,0)\} \;:\; c \in \mathbb{F}_q^*\Big\}.$$

A line in $PG(q,4)$ (1-dimensional subspace) is the set of all points whose coordinates satisfy three linearly independent homogeneous linear equations

$$\begin{aligned} a_{1,-2}x_{-2} + \;\cdots\; + a_{1,2}x_2 &= 0\\ a_{2,-2}x_{-2} + \;\cdots\; + a_{2,2}x_2 &= 0\\ a_{3,-2}x_{-2} + \;\cdots\; + a_{3,2}x_2 &= 0\,, \end{aligned}$$

with coefficients $a_{ij} \in \mathbb{F}_q$.

Let Q_4 be a quadratic surface defined by $Q_4 = \{x \in \mathcal{P} \;:\; x_0^2 + x_1x_{-1} + x_2x_{-2} = 0\}$.

Prove that the following properties hold:

(i) $|Q_4| = q^3+q^2+q+1$,

(ii) Q_4 contains q^3+q^2+q+1 lines (i.e. these are lines such that every point of the line lies in Q_4),

(iii) every line of Q_4 contains $q+1$ points,

(iv) every point is contained in $q+1$ lines of Q_4.