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25.05.07

Extremal Combinatorics — *SS 07*

Exercise Set 7

Let $\mathcal{P}$ be a set of *points*, and let $\mathcal{L}$ be a set of *lines*. A subset $I \subseteq \mathcal{P} \times \mathcal{L}$ is called an *incidence relation* on $(\mathcal{P}, \mathcal{L})$, and then we say that the triple $(\mathcal{P}, \mathcal{L}, I)$ is a *(rank two) geometry.*

The *incidence graph* Γ of the geometry is defined on $V(\Gamma) = \mathcal{P} \cup \mathcal{L}$, and its edge set is

$$E(\Gamma) = \{pl : p \in \mathcal{P}, l \in \mathcal{L}, (p, l) \in I\}.$$

We say that the function $\pi : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ is a *polarity* of the geometry $(\mathcal{P}, \mathcal{L}, I)$, if

(i) $\mathcal{P}^\pi = \mathcal{L}$ and $\mathcal{L}^\pi = \mathcal{P}$,

(ii) for every point p and every line l, we have that p and l are incident if and only if their polar images are incident, i.e $(p, l) \in I \iff (l^\pi, p^\pi) \in I$,

(iii) $\pi^2 = \text{id}$.

Observe that any polarity π is an automorphism of the incidence graph Γ.

In case a polarity exists we can define the polarity graph of the geometry which, compared to the incidence graph, cuts the number of vertices in half while keeps almost all degrees intact. Formally, the *polarity graph* Γ^π is the graph with $V(\Gamma^\pi) = \mathcal{P}$, and $E(\Gamma^\pi) = \{p_1p_2 : p_1 \neq p_2, (p_1, p_2^\pi) \in I\}$. We say that $p \in \mathcal{P}$ is an *absolute point* of π, if $(p, p^\pi) \in I$. Absolute points are the ones which would define a loop in the polarity graph, had we not excluded that possibility. The set of all absolute points is denoted by $N_\pi = \{p : \ p \text{ is absolute point of } \pi\}$.

Exercise 1

Prove Theorem 3.10 which states

(i) If $p \in N_\pi$, then we have $d_{\Gamma^\pi}(p) = d_\Gamma(p) - 1$. Otherwise, $d_{\Gamma^\pi}(p) = d_\Gamma(p)$.

(ii) $|V(\Gamma^\pi)| = \frac{1}{2}|V(\Gamma)|$ and $|E(\Gamma^\pi)| = \frac{1}{2}(|E(\Gamma)| - |N_\pi|)$.

(iii) If $C_{2k+1} \subseteq \Gamma^\pi$, then $C_{4k+2} \subseteq \Gamma$.

(iv) If $C_{2k} \subseteq \Gamma^\pi$, then there are two copies of C_{2k} in Γ such that one is the polar dual of the other.

(v) $g(\Gamma^\pi) \geq \frac{1}{2}g(\Gamma)$.

Exercise 2

Define

$$\mathcal{P}_* = \{(a, b, c) : a, b, c \in \mathbb{F}_q\}, \quad \mathcal{L}_* = \{[d, e, f] : d, e, f \in \mathbb{F}_q\}$$
$$((a, b, c), [d, e, f]) \in I_* \iff e - b = da, f - c = ea\,.$$

Prove that the incidence graph Γ_* of the above geometry is nothing else but Wenger's C_6-free construction.

Exercise 3

Let $q = 2^{2\alpha+1}$ and define π_* by

$$\begin{aligned} \pi_* : (a,b,c) &\mapsto \left[a^{2^{\alpha+1}}, (ab)^{2^\alpha} + c^{2^\alpha}, b^{2^{\alpha+1}}\right], \\ \pi_* : [d,e,f] &\mapsto \left(d^{2^\alpha}, f^{2^\alpha}, (df)^{2^\alpha} + e^{2^{\alpha+1}}\right). \end{aligned}$$

(a) Prove that π_* is a polarity of the above geometry $\Gamma_* = (\mathcal{P}_*, \mathcal{L}_*, I_*)$.

(b) Prove that the set of absolute points consists of

$$N_{\pi_*} := \left\{(a, b, a^{2^{\alpha+1}+2} + ab + b^{2^{\alpha+1}}) : a, b \in \mathbb{F}_q\right\}.$$

(c) Show that $\Gamma_*^{\pi_*}$ is C_6-free and conclude that $ex(n, C_6) \geq \frac{1}{2}n^{4/3} - \frac{1}{2}n^{2/3}$.