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Extremal Combinatorics — *SS 07*
Exercise Set 9

Let $(G, +)$ be a finite abelian group. A homomorphisms χ from $(G, +)$ into $(\mathbb{C}^*, \cdot)$ is called *character*, i.e. $\chi : G \to \mathbb{C}^*$ is a character of G if

$$\chi(a + b) = \chi(a)\chi(b), \ \forall a, b \in G.$$

From this definition we have that $\chi(a)$ is an n^{th} root of unity, and in particular $\chi(-a) = \chi(a)^{-1} = \overline{\chi(a)}$. The set of characters is denoted by $\hat{G}$.

Exercise 1

Prove that $\hat{G}$ is an abelian group with the following operation

$$(\chi\psi)(a) := \chi(a)\psi(a).$$

Exercise 2

Let ω be a primitive n^{th} root of unity and define the map $\chi_j : \mathbb{Z}_n \to \mathbb{C}^*$ by $\chi_j(a) := \omega^{ja}$.

Show that $\chi_j \in \hat{\mathbb{Z}}_n$ for every $j \in \mathbb{Z}$. Moreover, prove that the mapping $s : \mathbb{Z}_n \to \hat{\mathbb{Z}}_n$ defined by $s(j) = \chi_j$ is an isomorphism between $\mathbb{Z}_n$ and $\hat{\mathbb{Z}}_n$, that is $\mathbb{Z}_n \cong \hat{\mathbb{Z}}_n$.

Exercise 3

Show that if $G = H_1 \oplus H_2$, then $\hat{G} \cong \hat{H}_1 \oplus \hat{H}_2$.

Use this statement together with Exercise 2 to prove that $G \cong \hat{G}$.

Exercise 4

Let n be the number of citizen of Switzerland and note that the federal council of Switzerland consist of 7 people (they are called the Bundesrat). Assume the following (imaginary) voting scheme: Each person votes 1 (yes) or 0 (no) and the result of the whole vote is 1 if and only the whole federal council votes 1; otherwise it is 0. Formally

$$f := \{0,1\}^n \to \{0,1\}, \quad f(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \ldots \wedge x_7 .$$

Write f as a sum of characters $\chi_y = (-1)^{\langle y, \cdot \rangle}$, i.e. determine $\hat{f}(y)$ such that

$$f(x) = \sum_{y \in \{0,1\}^n} \hat{f}(y)(-1)^{\langle y, x \rangle} .$$