Projective Planes, Spaces

Projective Plane

Loosely speaking, a projective plane consists of a set of points and a set of lines together with an incidence relation determining whether a given point is on a given line or not.

Definition (Projective Plane): A *projective plane* Π is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is a set of elements called *points*, $\mathcal{L}$ is a subset of $2^{\mathcal{P}}$ called *lines*, and $\mathcal{I}$ is an *incidence relation* between points and lines such that the following holds:

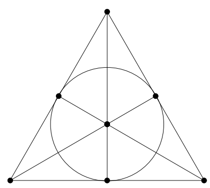
(i) every pair of distinct lines is incident to a unique point,

(ii) every pair of distinct points is incident to a unique line,

(iii) there are four points such that no three of them are incident to a single line.

A projective plane is called finite, if $\mathcal{P}$ is a finite set.

Example. The Fano plane is the smallest (nontrivial) example with seven points and seven lines, cf. picture. Note that the circle in the picture is also called a line. Every line contains three points.



One can show that there is a *duality* between points and lines (with respect to interchanging them in Axiom (iii)), from which the next result follows:

Theorem 1 *Let Π be a finite projective plane. Then there is an integer $m \geq 2$ such that every point (line, resp.) of Π is incident to exactly $m+1$ lines (points, resp.), and Π contains exactly $m^2 + m + 1$ points (lines, resp.).*

The integer m from the previous theorem is called *order* of the finite projective plane Π. We will be mainly interested in projective planes constructed algebraically via finite fields.

Remark: The prime power conjecture for projective planes (PPC) is a very famous conjecture saying that a projective plane of order m exists if and only if m is a prime power. It is known that there is no projective plane of order 6 and 10, where the latter has only recently be proved. However, the existence of projective plane of order 12 is still an open question. In the following we show the easy direction of PPC, that for every prime power there is a corresponding projective plane.

The Construction

For this purpose consider the finite q-element field $\mathbb{F}_q$. The projective plane $PG(q,2)$ over $\mathbb{F}_q$ is defined as follows:

- Points: The set of points $\mathcal{P}$ in $PG(q,2)$ are the equivalence classes of $\mathbb{F}_q^3 \backslash \{(0,0,0)\}$, where two triples are in relation if they are nonzero constant multiples of each other:

$$[x_0, x_1, x_2] = \left\{ (cx_0, cx_1, cx_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\} \ : \ c \in \mathbb{F}_q^* \right\} .$$

- Lines: Given a triple $(a_0, a_1, a_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$ we define the line $L(a_0, a_1, a_2)$ as follows:

$$L(a_0, a_1, a_2) := \left\{ [x_0, x_1, x_2] \in \mathcal{P} \ : \ a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \right\} .$$

  This also serves as the definition of the incidence relation. The set of lines $\mathcal{L}$ in $PG(q,2)$ consists of all the lines $L(a_0, a_1, a_2)$.

As an exercise verify the axioms of a projective plane and prove the according statements of Theorem 1 for $PG(q,2)$.

Projective Spaces

Analogs of the concept of projective planes can also be defined for dimensions higher than 2. However, we refrain from giving the abstract definition in favor of generalizing the example of projective planes over finite fields.

Consider the finite q-element field $\mathbb{F}_q$. Then the projective d-space over $\mathbb{F}_q$, denoted by $PG(q,d)$, is defined in the following way:

- Points: The set of points $\mathcal{P}$ in $PG(q,d)$ are the equivalence classes of $\mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\}$, where two $(d+1)$-tuples are in relation if they are nonzero constant multiples of each other:

$$[x_0, \ldots, x_d] = \left\{ (cx_0, \ldots, cx_d) \in \mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\} \ : \ c \in \mathbb{F}_q^* \right\} .$$

- A k-dimensional subspace in $PG(q,d)$ is the set of all points whose coordinates satisfy $d-k$ linearly independent homogeneous linear equations

$$\begin{array}{ccccccc} a_{10}x_0 & + & \cdots & + & a_{1d}x_d & = & 0 \\ \vdots & & \ddots & & \vdots & = & \vdots \\ a_{d-k,0}x_0 & + & \cdots & + & a_{d-k,d}x_d & = & 0 \, , \end{array}$$

  with coefficients $a_{ij} \in \mathbb{F}_q$.